IIE Transactions (2012) **44**, 834–867
Copyright © "IIE"
ISSN: 0740-817X print / 1545-8830 online
DOI: 10.1080/0740817X.2012.654846

The value of flexible backup suppliers and disruption risk information: newsvendor analysis with recourse

SOROUSH SAGHAFIAN and MARK P. VAN OYEN*

Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, MI 48109, USA
E-mail: soroush@umich.edu, vanoyen@umich.edu

Received July 2010 and accepted November 2011

This article develops a model and analysis to provide insight into two effective remedies to increase supply chain resilience: (i) contracting with a secondary flexible backup supplier; and (ii) monitoring primary suppliers to obtain disruption risk information. To investigate the true value of these strategies, an analysis is performed under imperfect information concerning the disruption risks and considering a two-stage setting with recourse. In this setting, the firm first monitors its suppliers and then utilizes a recourse option subject to the limited quantity of a capacity reserved *a priori* via a contract with a flexible backup supplier. The firm's jointly optimal behavior is analytically characterized (utilizing only the information available to the firm) regarding two interconnected decisions: (i) the advance capacity investment/reservation level with a flexible backup supplier; and (ii) the inventory ordering policy of the underlying products from both primary and backup suppliers. The presented results quantify effective disruption risk mitigation strategies for firms and provide managerial insights into the value of (i) a flexible backup supplier; (ii) disruption risk information; (iii) a contracted recourse option; and (iv) flexibility in the backup system.

Keywords: Flexible backup supplier, disruption risk information, disruption risk mitigation, backup capacity reservation

1. Introduction

In recent years, a variety of events have elevated to a strategic level concerns over the pernicious effects of supply chain disruptions. Consider the 2007 disruption in Boeing's 787 Dreamliner's supply chain. Advanced Integration Technology (AIT) fell months behind building parts needed to assemble the plane, thereby wreaking havoc upon Boeing's 787 inflexible supply chain. Boeing itself expected to take a cash hit of $2.5 billion in 2008 from paying penalties to airline customers and to keep its suppliers afloat in the wake of the serious cash flow disruption, according to Greising and Johnsson (2007). However, it became clear that AIT had in fact been facing serious production problems as early as 2006 (see, for instance, Greising and Johnsson (2007)). Thus, if Boeing had thoroughly monitored AIT in an effort to obtain *disruption risk information*, it might have made better ordering and contracting decisions in advance and thereby protected against such havoc. To enable firms to monitor their suppliers, some companies including Open Ratings have developed supply chain monitoring software that provides a firm with supplier visibility and actionable insights before a disruption occurs.

Consider also the disruption in the Toyota supply chain on February 1, 1997. A fire at the Aisin Seiki Co. destroyed most of its capacity to manufacture P-valves. Because of Aisin's ability to produce parts at low cost, Toyota had come to rely on Aisin for this product (Sheffi, 2007). According to the Wall Street Journal, Toyota officials called different part makers to obtain P-valves, including Somic (Reitman, 1997). Somic had the *flexibility* to free up machines and shift its production line to make P-valves. On February 6, right on schedule, it delivered its first P-valves to Toyota (Reitman, 1997). Considering the enormous financial impact of disruptions, it may be beneficial for many firms that procure raw materials from low-price, high-volume primary suppliers to reserve in advance some capacity from a *secondary flexible backup supplier* such as Somic to insure the supply stream against possible disruptions. This article gives insights into the potential benefits and risks.

Although disruptions are rare, their economic consequences can be massive. Hendricks and Singhal (2005) investigate over 800 cases of disruptions in supply chains and conclude that firms suffering from supply chain disruptions experience about 30% lower stock returns than their matched benchmarks (see also Kleindorfer *et al.* (2003)). Kleindorfer and Saad (2005) formulate a set of 10 principles derived from the supply chain risk literature, three of which are (i) *diversification* in sourcing; (ii) implementing *flexibility*; and (iii) *information* sharing. These principles

*Corresponding author

will be considered throughout this article from the perspective of a firm that procures materials from different sources.

We investigate two important strategic remedies to increase supply chain reliability and responsiveness: (i) contracting with a secondary flexible backup supplier (or similarly to establish an in-plant flexible resource) that is capable of adjusting its production mix to respond to the requests of a firm in the case of a disruption; and (ii) obtaining and assessing information about the disruption risk of primary suppliers. Point (i) increases the flexibility and responsiveness of supply chains in facing disruptions. In point (ii), monitoring suppliers allows firms to anticipate potential disruptions and adopt better operational decisions.

Using a model with two products, we quantify the value of purchasing flexible backup capacity through a *generalized capacity reservation* contract with a flexible backup supplier. In this contract, also known as an option contract, the buyer pays a fixed lump-sum payment to the supplier at the beginning of the contract in return for the delivery of any desired portion of the reserved fixed capacity at an additional proportional purchasing cost. (See Serel *et al.* (2001) for the traditional case that has only the lump-sum cost.) If the buyer has enough on-hand stock, less than the reserved capacity may be ordered to avoid additional holding and purchasing costs. Indeed, through this contract the buyer initially buys the option to order (any combination of the two products) up to a certain level from the supplier and later decides how to exercise this option by specifying the ordering quantity for each of the products. Moreover, based on the terms of the contract, the flexible backup supplier guarantees any amount of delivery up to the reserved capacity. Therefore, from the firm's point of view, this secondary supplier works as a reliable flexible backup (that can mitigate the risk of disruption in primary suppliers while reducing the cost of keeping excess inventory) with a limited flexible capacity in proportion to the up-front lump-sum investment.

Similar capacity reservation arrangements designed to provide the buyer with flexibility in the order quantity are observed in different high-tech industries such as semiconductors, consumer electronics, telecommunications, and pharmaceuticals (where the demand for high-tech products is highly volatile and difficult to forecast) as well as in some automakers (Henig *et al.*, 1997) and the textile and garment industry (Eppen and Iyer, 1997). Capacity reservation is also regarded as one of the countermeasures of the "bullwhip effect" (Lee *et al.*, 1997).

In addition to providing insights into purchasing flexible backup capacity (through a capacity reservation contract), we investigate the value of the firm's monitoring of unreliable suppliers to obtain a more accurate perception of disruption risks. Our analyses quantify the financial impact of the misperception of the true reliabilities of suppliers.

To perform our analysis, we consider a two-stage setting where the firm is endowed with a recourse possibility to selectively utilize the capacity reserved with the secondary flexible backup supplier *after* monitoring the risk of primary suppliers. Using our two-stage setting, we also investigate the value of implementing flexibility in the backup system. The value of information on disruption risks is computed, optimal ordering decisions are identified, and the optimal capacity reservation level is quantified (including bounds).

The remainder of the article is organized as follows. We review the literature in Section 2 and then in Section 3 we describe our model. Section 4 considers the problem in a two-stage setting with recourse. To gain insights into the value of recourse, in Section 5 we provide some benchmark analysis by considering the case where benchmark is not allowed. In Section 6, we use our framework to provide insights into the value of recourse, backup flexibility, and disruption risk information. Finally, we summarize our main findings in Section 7 and conclude.

2. Related literature

This article considers *all-or-nothing* supply: when the supplier is up it delivers an order in full, while nothing can be supplied when it is down. By contrast, in models with *yield uncertainty* or *random yield*, the quantity received is a random fraction of the quantity ordered (see Yano and Lee (1995) for an comprehensive review of the literature).

The majority of supply disruption papers focus on a single-supplier problem (see, for instance, Meyer *et al.* (1979), Bielecki and Kumar (1988), Parlar and Berkin (1991), Parlar and Perry (1995), Gupta (1996), Song and Zipkin (1996), Moinzadeh and Aggarwal (1997), Parlar (1997) and Arreola-Risa and De Croix (1998)). Parlar and Perry (1996) and Gürler and Parlar (1997) are among the supply disruption papers that consider more than one supplier. Both papers consider a firm that faces constant demand and sources from two identical-cost, infinite-capacity suppliers. Anupindi and Akella (1993) study a finite-horizon, discrete-time, independent and identically distributed stochastic continuous demand model in which there are two-zero-lead time random-yield suppliers.

More recent related supply disruption papers with multiple retailers include Babich (2006), Tomlin (2006), Tomlin and Snyder (2006), Babich *et al.* (2007), Chopra *et al.* (2007), Dada *et al.* (2007), Federgruen and Yang (2009), and Wang *et al.* (2010). Babich *et al.* (2007) study the effects of disruption risk in a supply chain where one retailer deals with competing risky suppliers who may default during their production lead times. Babich (2006) uses a single-period model of a two-echelon supply chain with competing risky suppliers and a single firm and investigates how the supplier default risk and default co-dependence affect firm procurement and production decisions. Tomlin (2006) sheds light on some effective disruption risk mitigation mechanism by considering a single-product setting in which a firm can source from two suppliers, one that is unreliable and another that is completely reliable but more expensive and may possess volume flexibility. Tomlin and Snyder (2006) investigate the value of a threat advisory system and develop multi-period models in which the firm has a single unreliable supplier, as well as models in which a second, perfectly reliable supplier is available. Dada *et al.* (2007) consider a newsvendor that is served by multiple suppliers, where any given supplier is identified to be either perfectly reliable or unreliable. Chopra *et al.* (2007) consider the effect of decoupling delays (recurrent risk) and disruption risks in a model with two suppliers: an unreliable supplier and a perfectly reliable supplier that is under a capacity reservation contract. Federgruen and Yang (2009) consider supply diversification under general supply risk for a single product and a single demand season. Wang *et al.* (2010) use two-stage stochastic programming settings and investigate the value of two disruption mitigation mechanisms: dual sourcing and process improvement.

The effect of disruption information has also been studied under different settings in some recent papers (see, for example, Tomlin (2009), Yang *et al.* (2009), and the references therein).

All of the above-mentioned papers examine a single product setting. However, a single-product setting does not allow us to capture the mix flexibility of a backup supplier such as Somic. To the best of our knowledge, this is the first article that develops a two-product analysis to simultaneously consider the option of implementing mix flexibility in supply and the possibility of obtaining disruption risk information.

For reviews of flexibility, we refer readers to Sethi and Sethi (1990), Gerwin (1993), and Suarez *et al.* (1995). In fact, the mix flexibility of production operations is studied in many papers, including Fine and Freund (1990), Jordan and Graves (1995), Kouvelis and Vairaktarakis (1998), Van Mieghem (1998), Graves and Tomlin (2003), Iravani *et al.* (2005), Tomlin and Wang (2005), and Iravani *et al.* (2011). Our study contributes to this literature by considering the value of a flexible supplier/resource to compensate for the unreliability of dedicated suppliers. Similar contributions, but in the context of the design of queueing systems with flexible resources, can be found in Saghafian *et al.* (2011) and the references therein.

3. Model and notation

Consider a centralized model of the contracting and ordering decisions of a firm (a manufacturer or retailer) in a two-echelon make-to-stock supply chain that produces/sells two types of products (namely, 1 and 2) and has two dedicated suppliers, each capable of supplying units (components, final products, or raw materials) for one of the products. Denote the dedicated supplier of units of product j as supplier j (for $j = 1, 2$). The firm also has a flexible backup supplier (namely f) that can produce (under a capacity reservation contract) units for both products 1 and 2, the sum of which cannot exceed a reserved capacity, $\bar{Q}^{\mathrm{f}}$. The capacity reservation contract explicitly allows the purchase of a flexible backup capacity, $\bar{Q}^{\mathrm{f}}$. We typically use subscripts for products, superscripts for suppliers, and employ the following notation:

h_j	:	holding cost per unit of product j,	$(j = 1, 2)$;
p_j	:	lost sale penalty cost per unit of unmet demand of product j,	$(j = 1, 2)$;
r_j	:	revenue per unit of product j sold,	$(j = 1, 2)$;
c^j	:	per unit purchasing cost of product j from dedicated supplier j,	$(j = 1, 2)$;
c_j^{f}	:	per unit purchasing cost of product j from flexible supplier,	$(j = 1, 2)$;
u^{f}	:	per unit capacity reservation cost of the flexible backup supplier,	
u_j^{f}	:	$(= u^{\mathrm{f}} + c_j^{\mathrm{f}})$ unit cost of product j from the flexible backup supplier,	$(j = 1, 2)$;
q^j	:	order quantity from dedicated supplier j,	$(j = 1, 2)$;
q_j^{f}	:	order quantity from the flexible backup supplier for product j,	$(j = 1, 2)$;
$\bar{Q}^{\mathrm{f}}$	:	reserved capacity (in units) from the flexible backup supplier.	

Figure 1 depicts the two-echelon supply chain model. We assume the firm has a *price-only* contract with its unreliable dedicated suppliers; i.e., the firm pays c^j $(\leq r_j)$ per unit delivered by dedicated (and unreliable) supplier j. Moreover, it has a generalized capacity reservation contract with its flexible backup supplier; i.e., the firm chooses the capacity level of $\bar{Q}^{\mathrm{f}}$ units and pays the flexible backup supplier a cost of $u^{\mathrm{f}} \times \bar{Q}^{\mathrm{f}}$ at time 0 to reserve its capacity. The flexible backup supplier, in return, agrees to deliver any order of products 1 and 2 (q_1^{f} and q_2^{f}) subject to $q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \leq \bar{Q}^{\mathrm{f}}$, and the firm pays the purchasing cost of $c_1^{\mathrm{f}} q_1^{\mathrm{f}} + c_2^{\mathrm{f}} q_2^{\mathrm{f}}$. We allow $u_j^{\mathrm{f}} = u^{\mathrm{f}} + c_j^{\mathrm{f}}$ to be greater or less than r_j. However, to avoid the trivial case where single sourcing from the flexible backup supplier is always optimal, we shall assume $u_j^{\mathrm{f}} = u^{\mathrm{f}} + c_j^{\mathrm{f}} > c^j$. This corresponds to practice, because the contract can be regarded as an investment in the flexible supplier's technology that is not cheaper than that of inflexible ones. We also note that the flexible supplier in our framework could be an in-plant flexible resource that requires an upfront investment of u^{f} per unit of capacity.

$$\text{For } j (j = 1, 2),\ \text{let } \mathcal{L}_j(x) = h_j[x]^+ + p_j[-x]^+ \text{ and}$$
$$L_j(x) = E_{D_j}[\mathcal{L}_j(x - D_j)] = h_j \int_0^x (x - \xi)\mathrm{d}F_j(\xi) + p_j \int_x^\infty (\xi - x)\mathrm{d}F_j(\xi), \tag{1}$$

where $[x]^+ = \max\{0, x\}$, and $F_j(\cdot)$ is the Cumulative Distribution Function (CDF) of the demand for product j

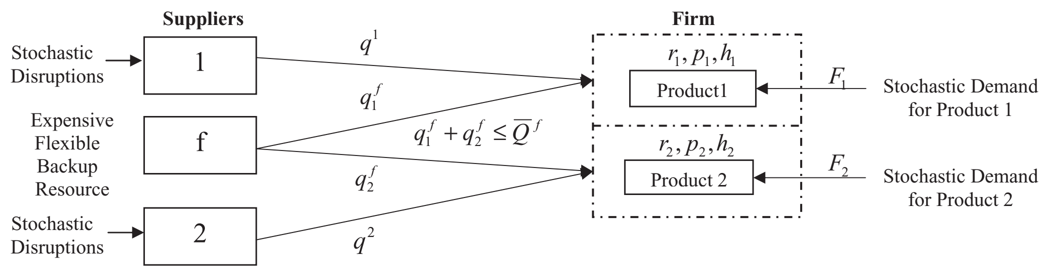


Fig. 1. The two-echelon supply chain under consideration.

(random variable D_j). We assume that $F_j(\cdot)$ (for both $j = 1, 2$) is differentiable and has $F_j(x) = 0$ for all $x \le 0$ and $F_j(x) > 0$ for all $x > 0$. Denoting the survival function (i.e., the complementary CDF) of demand of product j by $\bar{F}_j(\cdot) = 1 - F_j(\cdot)$, we define the inventory cost function $G_j(\cdot)$ as

$$G_j(x) = L_j(x) - r_j\ E[\min(D_j\ ,\ x)] = L_j(x) - r_j \int_0^x \bar{F}_j(\xi)\mathrm{d}\xi, \tag{2}$$

where the last expression can be obtained via integration by parts. In these definitions, $L_j(\cdot)$ is the expected total inventory cost of product j (holding plus shortage) and $G_j(\cdot)$ is the expected cost minus the expected revenue obtained from product j.

We note that a firm may not accurately perceive the disruption risk of its unreliable suppliers. However, we can model the firm's best estimate of the reliability of supplier j (i.e., the probability that supplier j is up) by θ^j. We define $\mathbf{\Theta} = (\theta^1, \theta^2)$ as the vector of *perceived* reliabilities and $\mathbf{\Pi} = (1 - \pi_0^1, 1 - \pi_0^2)$ as the vector of *true* reliabilities. Also, we let $\mathbf{\Upsilon} = (\epsilon^1, \epsilon^2)$ denote the firm's error in estimating the true reliability vector where $\mathbf{\Theta} = \mathbf{\Pi} + \mathbf{\Upsilon}$ (i.e., $\theta^j = 1 - \pi_0^j + \epsilon^j$ for both $j = 1, 2$).

4. Analyses with recourse (two-stage setting)

To generate insights into effective disruption mitigation mechanisms for firms, we start by analyzing the case where the firm can first monitor the "availability" (i.e., up or down) state of its primary suppliers and then utilize a recourse option of ordering from the secondary flexible backup supplier. Note that the availability state may be construed as the ability of a supplier to deliver the requested products within a required time. In other words, one can also think of the up (down) state in terms of on-time (late) delivery.

The sequence of events in this two-stage scenario is as follows. The firm first decides to reserve a capacity of $\bar{Q}^{\mathrm{f}}$ units from the secondary flexible backup supplier and pays $u^{\mathrm{f}} \times \bar{Q}^{\mathrm{f}}$ to do so (Stage 1). Then, the firm observes the state of its primary suppliers, purchases from the available ones, and uses the flexible backup supplier subject to the reserved capacity (Stage 2). Then, demands are realized and inventory costs (inventory shortage or holding minus the sales revenue) accrue. Observe that in Stage 1, the firm can insure the supply stream against possible disruptions through investment in a flexible backup capacity. It can then monitor the suppliers and use this information to make better ordering decisions. Indeed, in Stage 1, the firm can purchase the (recourse) option to benefit from flexible backup capacity proportional to its investment. In Stage 2, after observing the disruption states, the firm can exercise this option at a cost of c_j^{f} per mix of type j. After analyzing this sequence of events in Section 4.3, we will introduce an alternative sequence of events to allow consideration of offshore unreliable suppliers.

Using the above-mentioned framework we seek to answer the following questions.

Question 1: If the reserved capacity in Stage 1 is limited, how would the firm *distribute* the available flexible backup capacity among its products based on the obtained information?

Question 2: How much would a firm invest in the flexible supplier as a backup for possible disruptions? Does obtaining a recourse option result in a reduction in such an investment?

Also, comparing the scenario with recourse to the benchmark setting described in Section 5 (i.e., when the firm cannot observe the states of unreliable suppliers before ordering), we can ask the following question.

Question 3: How beneficial is having a recourse option for firms, and can it be regarded as a strong risk mitigation mechanism?

Moreover, it is interesting to compare a scenario with two dedicated backup suppliers (one for each product) with the one with a single but flexible backup supplier to answer the following question.

Question 4: How beneficial is the flexibility of a backup supplier, and for what firms should implementing flexibility (in the backup system) be more attractive?

Finally, similar to our no-recourse setting, we want to investigate the value of disruption risk information (that can reduce the risk belief errors) and address the following question.

Question 5: How valuable is obtaining disruption risk information (under a recourse option) for firms, and can it be regarded as a strong risk mitigation mechanism?

To answer such questions, let $C_{\mathrm{U,U}}(\bar{Q}^{\mathrm{f}})$, $C_{\mathrm{U,D}}(\bar{Q}^{\mathrm{f}})$, $C_{\mathrm{D,U}}(\bar{Q}^{\mathrm{f}})$, and $C_{\mathrm{D,D}}(\bar{Q}^{\mathrm{f}})$ (U: Up, D: Down) denote the minimum expected cost of the firm in the second stage, if both suppliers are up, only the first supplier is up, only the second supplier is up, and when none of them are up, respectively. These costs can be computed as follows.

$$C_{\mathrm{U,U}}(\bar{Q}^{\mathrm{f}}) = \min_{q^1,q^2,q_1^{\mathrm{f}},q_2^{\mathrm{f}}\geq 0 \text{ s.t. } q_1^{\mathrm{f}}+q_2^{\mathrm{f}}\leq \bar{Q}^{\mathrm{f}}} \sum_{j=1}^{2} c_j^{\mathrm{f}}\, q_j^{\mathrm{f}} + c^1\, q^1 + c^2\, q^2 + G_1\big(q_1 + q_1^{\mathrm{f}}\big) + G_2\big(q_2 + q_2^{\mathrm{f}}\big), \tag{3}$$

$$C_{\mathrm{U,D}}(\bar{Q}^{\mathrm{f}}) = \min_{q^1,q_1^{\mathrm{f}},q_2^{\mathrm{f}}\geq 0 \text{ s.t. } q_1^{\mathrm{f}}+q_2^{\mathrm{f}}\leq \bar{Q}^{\mathrm{f}}} \sum_{j=1}^{2} c_j^{\mathrm{f}}\, q_j^{\mathrm{f}} + \; c^1\, q^1 + G_1\big(q_1 + q_1^{\mathrm{f}}\big) + G_2\big(q_2^{\mathrm{f}}\big), \tag{4}$$

$$C_{\mathrm{D,U}}(\bar{Q}^{\mathrm{f}}) = \min_{q^2,q_1^{\mathrm{f}},q_2^{\mathrm{f}}\geq 0 \text{ s.t. } q_1^{\mathrm{f}}+q_2^{\mathrm{f}}\leq \bar{Q}^{\mathrm{f}}} \sum_{j=1}^{2} c_j^{\mathrm{f}}\, q_j^{\mathrm{f}} + \; c^2\, q^2 + G_1\big(q_1^{\mathrm{f}}\big) + G_2\big(q_2 + q_2^{\mathrm{f}}\big) \tag{5}$$

$$C_{\mathrm{D,D}}(\bar{Q}^{\mathrm{f}}) = \min_{q_1^{\mathrm{f}},q_2^{\mathrm{f}}\geq 0 \text{ s.t. } q_1^{\mathrm{f}}+q_2^{\mathrm{f}}\leq \bar{Q}^{\mathrm{f}}} \sum_{j=1}^{2} c_j^{\mathrm{f}}\, q_j^{\mathrm{f}} + G_1\big(q_1^{\mathrm{f}}\big) + G_2\big(q_2^{\mathrm{f}}\big). \tag{6}$$

Now, if $C_{\mathrm{Stage}\,2}(\bar{Q}^{\mathrm{f}})$ represents the optimal expected cost of Stage 2 as is perceived by the firm at the beginning of Stage 1, we have:

$$C_{\mathrm{Stage}\,2}(\bar{Q}^{\mathrm{f}}) = \theta^1\, \theta^2\, C_{\mathrm{U,U}}(\bar{Q}^{\mathrm{f}}) + \theta^1\, (1-\theta^2\,)C_{\mathrm{U,D}}(\bar{Q}^{\mathrm{f}}) + (1-\theta^1), \theta^2 C_{\mathrm{D,U}}(\bar{Q}^{\mathrm{f}}) + (1-\theta^1) \times (1-\theta^2)\, C_{\mathrm{D,D}}(\bar{Q}^{\mathrm{f}}). \tag{7}$$

Then, the firm can determine the optimal capacity reservation level ($\bar{Q}^{\mathrm{f}*}$) at the beginning of Stage 1 by solving the following program:

$$\min_{\bar{Q}^{\mathrm{f}}\geq 0} \quad u^{\mathrm{f}}\, \bar{Q}^{\mathrm{f}} + C_{\mathrm{Stage}\,2}(\bar{Q}^{\mathrm{f}}). \tag{8}$$

To determine the behavior of the firm, we must first optimize non-linear programs (3) to (6) of Stage 2 for a given capacity reservation level; thereafter, the optimal ordering policy can be used to derive the optimal contracting level as is perceived by the firm by solving program (8). This optimal capacity reservation level (a strategic decision) then determines the tactical ordering behavior in each case (i.e., the minimizers of programs (3) to (6)). To solve programs (3) to (6), we note that the objective functions are all jointly convex in their variables (see Appendix C for the proof of Lemma A1 and some other related results), and the constraints are linear. Hence, the Karush-Kuhn-Tuckor (KKT) conditions are sufficient and necessary to characterize the optimal solutions. To gain some insights, we first start by considering a single-product setting as a special case and next solve programs (3) to (6) to derive the optimal policy of the firm under the original two-product setting.

4.1. *Single–product special case*

Consider a single-product version of the problem discussed in the previous section. Since there is only one product and one unreliable supplier, we suppress both the product index, j, and the index of unreliable suppliers. However, we continue to use index f to denote the backup supplier. To characterize the firm's optimal capacity reservation level from the backup supplier as well as its optimal ordering policy, we need to first solve the problem in Stage 2 (i.e., derive the firm's optimal ordering policy) for any level of capacity reserved with the backup supplier in Stage 1. Subsequently, we can use the obtained results to solve Stage 1 to find the optimal capacity reservation level.

Proposition 1. (*Single product*). *Given that the firm reserves $\bar{Q}^{\mathrm{f}}$ units of capacity from the backup supplier in Stage* 1, *the optimal ordering policy of Stage* 2 *is as follows.*

1. *If $c^{\mathrm{f}} > c$ and the unreliable supplier is observed to be up, then $q^{\mathrm{f}*} = 0$, and $q^* = F^{-1}(p+r-c)/(p+r+h)$.*

2. *If $c^{\mathrm{f}} > c$ and the unreliable supplier is observed to be down, then $q^{\mathrm{f}*} = \min\{F^{-1}(p+r-c^{\mathrm{f}})/(p+r+h),\, \bar{Q}^{\mathrm{f}}\}$.*

3. *If $c^{\mathrm{f}} \leq c$ and the unreliable supplier is observed to be up, then:*

   (i) *If*
   $$\bar{Q}^{\mathrm{f}} \in \left[\, F^{-1}\left(\frac{p+r-c^{\mathrm{f}}}{p+r+h}\right), +\infty \,\right],$$
   then
   $$q^{\mathrm{f}*} = F^{-1}\left(\frac{p+r-c^{\mathrm{f}}}{p+r+h}\right) \textit{ and } q^* = 0.$$

   (ii) *If*
   $$\bar{Q}^{\mathrm{f}} \in \left[\, F^{-1}\left(\frac{p+r-c}{p+r+h}\right), F^{-1}\left(\frac{p+r-c^{\mathrm{f}}}{p+r+h}\right) \,\right],$$
   then
   $$q^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}} \;\textit{ and }\; q^* = 0.$$

   (iii) *If*
   $$\bar{Q}^{\mathrm{f}} \in \left[\, 0,\; F^{-1}\left(\frac{p+r-c}{p+r+h}\right) \,\right],$$
   then
   $$q^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}} \textit{ and } q^* = F^{-1}\left(\frac{p+r-c}{p+r+h}\right) - \bar{Q}^{\mathrm{f}}.$$

4. *If $c^{\mathrm{f}} \le c$ and the unreliable supplier is observed to be down, then:*
   (i) *If*

$$\bar{Q}^{\mathrm{f}} \in \left[F^{-1}\left(\frac{p+r-c^{\mathrm{f}}}{p+r+h}\right), +\infty \right],$$

then

$$q^{\mathrm{f}*} = F^{-1}\left(\frac{p+r-c^{\mathrm{f}}}{p+r+h}\right).$$

   (ii) *If*

$$\bar{Q}^{\mathrm{f}} \in \left[0,\ F^{-1}\left(\frac{p+r-c^{\mathrm{f}}}{p+r+h}\right) \right],$$

then

$$q^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}}.$$

Proof. All proofs are provided in Appendix A. ■

Parts 1 and 2 of Proposition 1 describe that when exercising the option from the backup supplier is more expensive than purchasing from the unreliable supplier, the firm orders only from a single source: it only uses the unreliable supplier if it is up and only uses the backup supplier (as much as required or the reserved capacity allows) otherwise. As parts 3 and 4 of Proposition 1 show, when exercising the option from the backup supplier is cheaper than purchasing from the unreliable supplier, if the unreliable supplier is observed to be up, then the firm either orders nothing from the unreliable supplier or exhausts the capacity reservation option (unless the reserved capacity is sufficient) and only orders the rest of its requirement from the unreliable supplier. On the other hand, when the unreliable supplier is observed to be down, the firm exhausts the capacity reservation option (unless the reserved capacity is sufficient). In fact, the firm's problem in a single-product setting is much easier than the two-product setting, since it is not facing the complex problem of rationing the available limited *flexible* backup capacity between the two products in an effective way. As we will show, in the two-product setting, even when only one of the suppliers is down, the firm may need to ration the backup flexible capacity between the two products (see Theorem 2 part (2)).

Remark 1. A special case of the single-product version of our model presented in this section is the case where the cost of reserving capacity in Stage 1 (u^{f}) is negligible. In this very special case where (i) the backup supplier is not endowed with mix flexibility (since it is a single-product setting) and (ii) backup capacity is unlimited (i.e., $\bar{Q}^{\mathrm{f}}$ is large enough) and does not need to be reserved in advance, the role of the backup supplier can be construed via the second opportunity quick response models studied thoroughly in papers such as Fisher and Raman (1996); (the Sport Obermeyer case), Eppen and Iyer (1997), Milner and Kouvelis (2002), and Li *et al.* (2009). The main difference between our work and such models, even under the special case of a single-product setting, is that the firm must optimize the amount of backup capacity to be reserved in advance (i.e., in anticipation of potential future disruptions) subject to a reservation cost. Indeed, the firm in our model can ensure a supply stream by buying backup capacity in advance. Note that this investment greatly affects the ordering ability of the firm in the second stage. Furthermore, another main feature of our model that differentiates it from such studies is the mix flexibility of the backup flexible supplier that we will consider in the following. In the presence of a backup capacity that is (i) limited and (ii) flexible, we provide insights into the question of how to effectively benefit from a recourse option and *ration* the limited backup capacity between products to compensate for the disruption risk of primary suppliers. Comparing our setting with a setting where the backup capacity is not flexible (i.e., a setting with two independent products), we will see that the mix flexibility in the backup system provides a significant advantage for the firm in the presence of unreliable suppliers. Another distinct and novel objective of our research is to provide insights into the value of obtaining disruption risk information as we will discuss in Sections 5.2 and 6.3.

Having the ordering policy in hand, we can now use Proposition 1 to determine the optimal capacity reservation level of Stage 1.

Proposition 2. (*Capacity reservation level*). *The optimal capacity reservation level* (*as perceived by the firm*) *can be characterized as follows*:

1. *If $c^{\mathrm{f}} > c$, then:*

$$\bar{Q}^{\mathrm{f}*} = F^{-1}\Big(\Big[\frac{p+r-((u^{\mathrm{f}}+(1-\theta)c^{\mathrm{f}})/(1-\theta))}{p+r+h}\Big]^{+}\Big).$$

2. *If $c^{\mathrm{f}} \le c$, then:*

$$\bar{Q}^{\mathrm{f}*} = F^{-1}\Big(\Big[\frac{p+r-((u^{\mathrm{f}}+c^{\mathrm{f}}-\theta c)/(1-\theta))}{p+r+h}\Big]^{+}\Big).$$

Parts 1 and 2 of Proposition 2 can be combined and summarized as follows. Let $\bar{c} = \max\{c, c^{\mathrm{f}}\}$ and $\hat{c} = (u^{\mathrm{f}} + c^{\mathrm{f}} - \theta\bar{c})/(1-\theta)$. Then,

$$\bar{Q}^{\mathrm{f}*} = F^{-1}\left(\left[\frac{p+r-\hat{c}}{p+r+h}\right]^{+}\right),$$

which can be construed as a single-source (and single-stage) traditional newsvendor setting with a perfectly reliable source and a purchasing cost $\hat{c}$. However, note that this new purchasing cost, $\hat{c}$, is affected by the reliability perception of the firm, θ, as well as the backup capacity reservation costs, u^{f} and c^{f}. For instance, the firm will not reserve any backup capacity in Stage 1 if its reliability perception of the unreliable supplier is greater than a threshold (that depends on the inventory, purchasing, and capacity reservation costs), even though it might be truly beneficial to reserve some backup capacity.

4.2. *Two-product case*

We now consider our original two-product setting to provide insights into Questions 1 to 5. The following three theorems solve programs (3) to (6) to define the optimal ordering policy of the firm and provide insight into Question 4. For brevity, we only consider the most interesting situation where $c_j^{\mathrm{f}} \le c^j$ throughout this section, but other situations can be analyzed in a similar way (see Appendix D for the case where $c_j^{\mathrm{f}} > c^j$).

Theorem 1. (*Both suppliers up*). *Let* $k = \operatorname{argmax}\{c^j - c_j^{\mathrm{f}} : j = 1, 2\}$ *denote the product with the higher difference in purchasing cost, and* $l = 3 - k$ *be the other product. If both suppliers are observed to be up, then the following cases fully characterize the optimal ordering policy of the firm, given a reserved flexible backup capacity of* $\bar{Q}^{\mathrm{f}}$.

1. *If*

$$\bar{Q}^{\mathrm{f}} \in \left[\sum_{j=k,l} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right), +\infty \right),$$

then

$$q_j^{\mathrm{f}*} = F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right) \quad \textit{and} \quad q^{j*} = 0 \ (j = k, l).$$

2. *If*

$$\bar{Q}^{\mathrm{f}} \in \left[\sum_{j=k,l} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}} - (c^l - c_l^{\mathrm{f}})}{p_j + r_j + h_j}\right), \right. \left. \sum_{j=k,l} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right) \right]$$

then

$$q_j^{\mathrm{f}*} = F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}} - t}{p_j + r_j + h_j}\right) \ \textit{and} \ q^{j*} = 0 \ (j = k, l),$$

where $t \in (0, c^l - c_l^{\mathrm{f}}]$ *is the solution to*

$$\sum_{j=k,l} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}} - t}{p_j + r_j + h_j}\right) = \bar{Q}^{\mathrm{f}}.$$

3. *If*

$$\bar{Q}^{\mathrm{f}} \in \left[F_k^{-1}\left(\frac{p_k + r_k - c_k^{\mathrm{f}} - (c^l - c_l^{\mathrm{f}})}{p_k + r_k + h_k}\right), \right. \left. \sum_{j=k,l} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}} - (c^l - c_l^{\mathrm{f}})}{p_j + r_j + h_j}\right) \right]$$

then

$$q_k^{\mathrm{f}*} = F_k^{-1}\left(\frac{p_k + r_k - c_k^{\mathrm{f}} - (c^l - c_l^{\mathrm{f}})}{p_k + r_k + h_k}\right), q_l^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}} - q_k^{\mathrm{f}*},$$

$$q^{k*} = 0,$$

and

$$q^{l*} = F_l^{-1}\left(\frac{p_l + r_l - c^l}{p_l + r_l + h_l}\right) - q_l^{\mathrm{f}*}.$$

4. *If*

$$\bar{Q}^{\mathrm{f}} \in \left[F_k^{-1}\left(\frac{p_k + r_k - c^k}{p_k + r_k + h_k}\right), \right. \left. F_k^{-1}\left(\frac{p_k + r_k - c_k^{\mathrm{f}} - (c^l - c_l^{\mathrm{f}})}{p_k + r_k + h_k}\right) \right],$$

then:

$$q_k^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}}, \quad q_l^{\mathrm{f}*} = 0, \quad q^{k*} = 0,$$

and

$$q^{l*} = F_l^{-1}\left(\frac{p_l + r_l - c^l}{p_l + r_l + h_l}\right).$$

5. *If*

$$\bar{Q}^{\mathrm{f}} \in \left[0, \ F_k^{-1}\left(\frac{p_k + r_k - c^k}{p_k + r_k + h_k}\right) \right],$$

then:

$$q_k^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}}, \ q_l^{\mathrm{f}*} = 0, \ q^{k*} = F_k^{-1}\left(\frac{p_k + r_k - c^k}{p_k + r_k + h_k}\right) - \bar{Q}^{\mathrm{f}},$$

and

$$q^{l*} = F_l^{-1}\left(\frac{p_l + r_l - c^l}{p_l + r_l + h_l}\right).$$

Theorem 1 part (1) shows that if the firm has already reserved more than enough capacity (in Stage 1), it would not order anything from the primary suppliers (in Stage 2) and will only use the available flexible capacity, ordering the optimal level for each of the products. The rest of the reserved capacity is wasted to avoid paying extra holding or purchasing costs. Even if the reserved capacity is below the level identified in part (1) but is in the range described by part (2), the firm will only use the reserved capacity. However, in this case, it will appropriately *ration* $\bar{Q}^{\mathrm{f}}$ between the products. Indeed, t can be viewed as a fictitious additional ordering cost that is applied to optimally ration the available flexible capacity. Part (3) states that if the available capacity is enough to fulfill a prescribed amount of product k but not enough for both of the products, the firm should use the flexible capacity to satisfy all of the optimal ordering amount of product k. Hence, it would not order anything from primary supplier k and use the rest of the reserved capacity as well as primary supplier l to satisfy the requirement for product l. In the case that the reserved capacity is not enough to meet the prescribed level for any of the products but still is relatively large, part (4) shows that the firm should set aside all the reserved flexible capacity for product k (and not order from dedicated supplier k). Hence, in this case, it is optimal to only use primary supplier l to optimize the service level of product l. If the reserved flexible capacity is very low, as is presented in part (5), it is optimal to use all $\bar{Q}^{\mathrm{f}}$ units of the limited flexible capacity for the "expensive" product (i.e., product k) and also procure the rest of requirements of this product from its primary supplier. Moreover, similar to case (6), product l is sourced only through its primary supplier. We now treat the case when exactly one of the suppliers is observed to be disrupted.

Theorem 2. (*One supplier up*) *Let* $m \in \{1, 2\}$ *denote the dedicated supplier that is observed to be up*, *and* $n = 3 - m$ *be the disrupted supplier. Then*, *the following cases fully characterize the optimal ordering policy of the firm*, *given a reserved flexible backup capacity of* $\bar{Q}^{\mathrm{f}}$:

1. *If*

$$\bar{Q}^{\mathrm{f}} \in \left[\sum_{j=n,m} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right), +\infty \right],$$

then:

$$q_j^{\mathrm{f}*} = F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right) (j = n, m) \ \textit{and} \ q^{m*} = 0.$$

2. *If*

$$\bar{Q}^{\mathrm{f}} \in \left[\sum_{j=n,m} F_j^{-1}\left(\left[\frac{p_j + r_j - c_j^{\mathrm{f}} - \left(c^m - c_m^{\mathrm{f}}\right)}{p_j + r_j + h_j}\right]^+\right), \sum_{j=n,m} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right) \right)$$

then:

$$q_j^{\mathrm{f}*} = F_j^{-1}\left(\left[\frac{p_j + r_j - c_j^{\mathrm{f}} - t}{p_j + r_j + h_j}\right]^+\right) (j = n, m) \ \textit{and} \ q^{m*} = 0,$$

where $t \in (0, c^m - c_m^{\mathrm{f}}]$ *is the solution to*

$$\sum_{j=n,m} F_j^{-1}\left(\left[\frac{p_j + r_j - c_j^{\mathrm{f}} - t}{p_j + r_j + h_j}\right]^+\right) = \bar{Q}^{\mathrm{f}}.$$

3. *If*

$$\bar{Q}^{\mathrm{f}} \in \left[F_n^{-1}\left(\left[\frac{p_n + r_n - c_n^{\mathrm{f}} - \left(c^m - c_m^{\mathrm{f}}\right)}{p_n + r_n + h_n}\right]^+\right), \sum_{j=n,m} F_j^{-1}\left(\left[\frac{p_j + r_j - c_j^{\mathrm{f}} - \left(c^m - c_m^{\mathrm{f}}\right)}{p_j + r_j + h_j}\right]^+\right) \right]$$

then:

$$q_n^{\mathrm{f}*} = F_n^{-1}\left(\left[\frac{p_n + r_n - c_n^{\mathrm{f}} - \left(c^m - c_m^{\mathrm{f}}\right)}{p_n + r_n + h_n}\right]^+\right), \quad q_m^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}} - q_n^{\mathrm{f}*},$$

and

$$q^{m*} = F_m^{-1}\left(\frac{p_m + r_m - c^m}{p_m + r_m + h_m}\right) - q_m^{\mathrm{f}*}.$$

4. *If*

$$\bar{Q}^{\mathrm{f}} \in \left[0, F_n^{-1}\left(\left[\frac{p_n + r_n - c_n^{\mathrm{f}} - \left(c^m - c_m^{\mathrm{f}}\right)}{p_n + r_n + h_n}\right]^+\right) \right]$$

then:

$$q_n^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}}, \quad q_m^{\mathrm{f}*} = 0,$$

and

$$q^{m*} = F_m^{-1}\left(\frac{p_m + r_m - c^m}{p_m + r_m + h_m}\right).$$

It is noteworthy that Theorem 2 can be interpreted via Theorem 1. In fact, by considering the disrupted supplier as a supplier with a sufficiently large purchasing cost, Theorem 2 coincides with Theorem 1. However, in Theorem 2 the operator $[\cdot]^+$ is used (whenever necessary) to ensure that the ratios are within appropriate domain of functions $F_j^{-1}(\cdot)$. This is redundant in Theorem 1 because of the assumption $r_j \geq c^j = c_j^{\mathrm{f}} + (c^j - c_j^{\mathrm{f}})$ (for both $j = k, l$), which also implies $r_k \geq c_k^{\mathrm{f}} + \left(c^l - c_l^{\mathrm{f}}\right)$ since $c^k - c_k^{\mathrm{f}} \geq c^l - c_l^{\mathrm{f}}$. To conclude, the following theorem treats the case when both primary suppliers are disrupted and similarly can be interpreted via Theorem 1 with sufficiently large c^j $(j = 1, 2)$.

Theorem 3. (*Both suppliers disrupted*). *The following cases fully characterize the optimal ordering policy of the firm when both primary suppliers are observed to be disrupted*, *given a reserved flexible backup capacity of* $\bar{Q}^{\mathrm{f}}$:

1. *If*

$$\bar{Q}^{\mathrm{f}} \in \left[\sum_{j=1,2} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right), +\infty \right],$$

then:

$$q_j^{\mathrm{f}*} = F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right) (j = 1, 2).$$

2. *If*

$$\bar{Q}^{\mathrm{f}} \in \left[0, \sum_{j=1,2} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right) \right],$$

then:

$$q_j^{\mathrm{f}*} = F_j^{-1}\left(\left[\frac{p_j + r_j - c_j^{\mathrm{f}} - t}{p_j + r_j + h_j}\right]^+\right) (j = 1, 2),$$

where $t \in (0, \max_j\{p_j + r_j - c_j^{\mathrm{f}}\}]$ *is the solution to*

$$\sum_{j=1,2} F_j^{-1}\left(\left[\frac{p_j + r_j - c_j^{\mathrm{f}} - t}{p_j + r_j + h_j}\right]^+\right) = \bar{Q}^{\mathrm{f}}.$$

Theorems 1, 2, and 3 solve programs (3) to (6). Note that the separability of the ordering policy between products (as reflected in Theorem 7) no longer applies. However, to answer Question 2, one can use Theorems 1, 2, and 3 to solve program (8) and obtain the optimal capacity reservation level (for any parameter setting and any demands distributions $F_1(\cdot)$, $F_2(\cdot)$). For brevity, and to further analytically characterize the optimal contracting level, however, we only consider the case where the following mild assumption holds. For simplicity of presentation, it is also convenient to fix the labeling of the products such that product 2 is the product with higher difference in purchasing cost (i.e., $\mathrm{argmax}\{c^j - c_j^{\mathrm{f}} : j = 1, 2\} = 2$).

Assumption 1. Demand distributions $F_1(\cdot)$, $F_2(\cdot)$ are such that:

(i)

$$0 \le F_1^{-1}\left(\frac{p_1 + r_1 - c_1^{\mathrm{f}} - (c^2 - c_2^{\mathrm{f}})}{p_1 + r_1 + h_1}\right) \le F_2^{-1}\left(\frac{p_2 + r_2 - c^2}{p_2 + r_2 + h_2}\right),$$

and

(ii)

$$F_2^{-1}\left(\frac{p_2 + r_2 - c_2^{\mathrm{f}} - (c^1 - c_1^{\mathrm{f}})}{p_2 + r_2 + h_2}\right) \le \sum_{j=1,2} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}} - (c^2 - c_2^{\mathrm{f}})}{p_j + r_j + h_j}\right).$$

Notice that the above assumption is not critical, since (i) if it does not hold, analysis will follow similar lines; and (ii) it holds for most settings where products are not extremely different in their procurement and inventory costs as well as their demand distributions. For instance, a completely symmetric scenario where the parameters are product independent satisfies conditions (i) and (ii).

The following proposition further helps to characterize the optimal capacity reservation level with the flexible backup supplier. It states that $\bar{q}^{\mathrm{f}*}$ can only take one of six possible values. Therefore, it can be found by comparing the cost of these six options.

Proposition 3. (*Capacity reservation level*). *Under Assumption* 1, $\bar{q}^{\mathrm{f}*} \in \{\bar{Q}_k^{\mathrm{f}} : k = 1, 2, \ldots, 6\}$, *where*:

$$\begin{aligned}
\bar{Q}_1^{\mathrm{f}} &\triangleq \operatorname*{arg\,min}_{\bar{Q}^{\mathrm{f}} \in I_1} \left(u^{\mathrm{f}} - \theta^1\theta^2(c^2 - c_2^{\mathrm{f}}) + \theta^1\bar{\theta}^2 c_2^{\mathrm{f}} + \bar{\theta}^1\theta^2 c_1^{\mathrm{f}}\right)\bar{Q}^{\mathrm{f}} \\
&\quad + \bar{\theta}^1\theta^2 G_1(\bar{Q}^{\mathrm{f}}) + \theta^1\bar{\theta}^2 G_2(\bar{Q}^{\mathrm{f}}) + \bar{\theta}^1\bar{\theta}^2\Gamma(\bar{Q}^{\mathrm{f}}), \\
\bar{Q}_2^{\mathrm{f}} &\triangleq \operatorname*{arg\,min}_{\bar{Q}^{\mathrm{f}} \in I_2} \left(u^{\mathrm{f}} - \theta^2(c^2 - c_2^{\mathrm{f}}) + \theta^1\bar{\theta}^2 c_2^{\mathrm{f}}\right)\bar{Q}^{\mathrm{f}} \\
&\quad + \theta^1\bar{\theta}^2 G_2(\bar{Q}^{\mathrm{f}}) + \bar{\theta}^1\bar{\theta}^2\Gamma(\bar{Q}^{\mathrm{f}}), \\
\bar{Q}_3^{\mathrm{f}} &\triangleq \operatorname*{arg\,min}_{\bar{Q}^{\mathrm{f}} \in I_3} \left(u^{\mathrm{f}} - \bar{\theta}^1\theta^2(c^2 - c_2^{\mathrm{f}}) + \theta^1 c_2^{\mathrm{f}}\right)\bar{Q}^{\mathrm{f}} \\
&\quad + \theta^1 G_2(\bar{Q}^{\mathrm{f}}) + \bar{\theta}^1\bar{\theta}^2\Gamma(\bar{Q}^{\mathrm{f}}), \\
\bar{Q}_4^{\mathrm{f}} &\triangleq \operatorname*{arg\,min}_{\bar{Q}^{\mathrm{f}} \in I_4} \left(u^{\mathrm{f}} - \theta^1(c^1 - c_1^{\mathrm{f}}) - \bar{\theta}^1\theta^2(c^2 - c_2^{\mathrm{f}})\right)\bar{Q}^{\mathrm{f}} \\
&\quad + \bar{\theta}^1\bar{\theta}^2\Gamma(\bar{Q}^{\mathrm{f}}), \\
\bar{Q}_5^{\mathrm{f}} &\triangleq \operatorname*{arg\,min}_{\bar{Q}^{\mathrm{f}} \in I_5} \left(u^{\mathrm{f}} - \theta^1(c^1 - c_1^{\mathrm{f}})\right)\bar{Q}^{\mathrm{f}} + \bar{\theta}^1\Gamma(\bar{Q}^{\mathrm{f}}), \\
\bar{Q}_6^{\mathrm{f}} &\triangleq \operatorname*{arg\,min}_{\bar{Q}^{\mathrm{f}} \in I_6} u^{\mathrm{f}}\bar{Q}^{\mathrm{f}} + \Gamma(\bar{Q}^{\mathrm{f}}),
\end{aligned}$$

with

$$\bar{\theta}^j = 1 - \theta^j, \Gamma(\bar{Q}^{\mathrm{f}}) = \sum_{j=1,2}\left[c_j^{\mathrm{f}} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}} - t_{\bar{Q}^{\mathrm{f}}}}{p_j + r_j + h_j}\right) + G_j(F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}} - t_{\bar{Q}^{\mathrm{f}}}}{p_j + r_j + h_j})\right)\right]$$

and $t_{\bar{Q}^{\mathrm{f}}}$ *is the solution to*

$$\sum_{j=1,2} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}} - t_{\bar{Q}^{\mathrm{f}}}}{p_j + r_j + h_j}\right) = \bar{Q}^{\mathrm{f}},$$

and

$$\begin{aligned}
I_1 &= \left[0, F_1^{-1}\left(\frac{p_1 + r_1 - c_1^{\mathrm{f}} - (c^2 - c_2^{\mathrm{f}})}{p_1 + r_1 + h_1}\right)\right], \\
I_2 &= \left[F_1^{-1}\left(\frac{p_1 + r_1 - c_1^{\mathrm{f}} - (c^2 - c_2^{\mathrm{f}})}{p_1 + r_1 + h_1}\right), F_2^{-1}\left(\frac{p_2 + r_2 - c^2}{p_2 + r_2 + h_2}\right)\right]. \\
I_3 &= \left[F_2^{-1}\left(\frac{p_2 + r_2 - c^2}{p_2 + r_2 + h_2}\right), F_2^{-1}\left(\frac{p_2 + r_2 - c_2^{\mathrm{f}} - (c^1 - c_1^{\mathrm{f}})}{p_2 + r_2 + h_2}\right)\right], \\
I_4 &= \left[F_2^{-1}\left(\frac{p_2 + r_2 - c_2^{\mathrm{f}} - (c^1 - c_1^{\mathrm{f}})}{p_2 + r_2 + h_2}\right), \sum_{j=1,2} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}} - (c^2 - c_2^{\mathrm{f}})}{p_j + r_j + h_j}\right)\right]
\end{aligned}$$

$$I_5 = \Bigg[\sum_{j=1,2} F_j^{-1}\Bigg(\frac{p_j + r_j - c_j^{\mathrm{f}} - \big(c^2 - c_2^{\mathrm{f}}\big)}{p_j + r_j + h_j}\Bigg), \sum_{j=1,2} F_j^{-1}\Bigg(\frac{p_j + r_j - c_j^{\mathrm{f}} - \big(c^1 - c_1^{\mathrm{f}}\big)}{p_j + r_j + h_j}\Bigg)\Bigg],$$

$$I_6 = \Bigg[\sum_{j=1,2} F_j^{-1}\Bigg(\frac{p_j + r_j - c_j^{\mathrm{f}} - \big(c^1 - c_1^{\mathrm{f}}\big)}{p_j + r_j + h_j}\Bigg), \sum_{j=1,2} F_j^{-1}\Bigg(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\Bigg)\Bigg].$$

We now present a corollary of the above proposition that provides two upper bounds on the optimal capacity reservation (or investment) level with the flexible backup supplier. The first upper bound is the sum of optimal orders to two separate, reliable, and dedicated suppliers with purchasing costs c_j^{f} $(j = 1, 2)$. This bound shows that the flexibility of the backup supplier offers competitive advantage to the firm. The second bound is the optimal capacity reservation level without the recourse option (see Section 5 for analysis without the recourse option). Indeed, the second part of the following corollary provides more insights to Question 2 by presenting a condition under which obtaining a recourse option will result in a non-strict reduction in the capacity reserved (or the investment level) with the flexible supplier.

Corollary 1. (*Bounds*). *The optimal capacity reservation level with the flexible backup supplier* (*with recourse*) *is bounded above by*

(i)

$$\sum_{j=1,2} F_j^{-1}\Bigg(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\Bigg),$$

and

(ii) *the optimal capacity reservation level without recourse, if disruption risks are such that* $\theta^j\,(c^j - c_j^{\mathrm{f}}) \geq u^{\mathrm{f}}$ *for* $j = 1, 2$.

To provide more insights, we now characterize the optimal backup capacity reservation level under a symmetric scenario where the two products have similar characteristics (demand distributions, procurement and inventory costs, and revenues—but not necessarily supplier reliabilities). Such a symmetric scenario allows us to completely characterize the optimal capacity reservation level and gain some clear insights. To this end, the following theorem considers a symmetric scenario and characterizes both cases where the flexible secondary supplier is more expensive or cheaper than the primary ones.

Theorem 4. (*Symmetric scenario*). *Consider a symmetric scenario where all parameters except* (*perhaps*) *the supplier reliabilities are product-independent. Further, assume that product demands follow a uniform distribution between 0 and* d (*with* $d > (2\,(p + r - u^{\mathrm{f}} - c^{\mathrm{f}}))/(p + r + h)$ *to ensure full linearity of the demand CDFs in the working range*). *Then, if* $c - c^{\mathrm{f}} \geq u^{\mathrm{f}}$, *we have*:

$$\bar{Q}^{\mathrm{f}*} = \frac{2d\,(p + r - u^{\mathrm{f}} - c^{\mathrm{f}})}{p + r + h}. \tag{9}$$

However, if $c - c^{\mathrm{f}} < u^{\mathrm{f}}$, *consider the following conditions on the reliability of suppliers*:
Condition 1 (C1): $\bar{\theta}^1\,\bar{\theta}^2 \geq 2\,(u^{\mathrm{f}} - (c - c^{\mathrm{f}}))/(p + r - c)$, *and Condition 2 (C2)*: $\theta^1\,\theta^2 \leq 1 - (u^{\mathrm{f}} - (c - c^{\mathrm{f}}))/(p + r - c)$. *Then*:

(i) *when both C1 and C2 do not hold*: $\bar{q}^{\mathrm{f}*} = 0$;
(ii) *when C1 does not hold but C2 holds*:

$$\bar{Q}^{\mathrm{f}*} = \frac{d\,((1 - \theta^1\theta^2)(p + r) + \theta^1\theta^2 c - u^{\mathrm{f}} - c^{\mathrm{f}})}{(\bar{\theta}^1\theta^2 + \theta^1\bar{\theta}^2 + (\bar{\theta}^1\bar{\theta}^2)/(2)(p + r + h)}, \tag{10}$$

(iii) *when C1 holds*:

$$\bar{Q}^{\mathrm{f}*} = \frac{2d\,(p + r - c - (u^{\mathrm{f}} - (c - c^{\mathrm{f}}))/(\bar{\theta}^1\,\bar{\theta}^2))}{p + r + h}. \tag{11}$$

The above theorem describes that, when the procurement cost from the flexible supplier $(u^{\mathrm{f}} + c^{\mathrm{f}})$ is sufficiently cheap, the optimal capacity reservation level is independent of disruption risks. Furthermore, similar to the case without recourse (see Theorem 7), there is a separation of the joint capacity reservation in this case; the optimal capacity reserved presented in Equation (9) is the sum of the orders of two independent newsvendors with a procurement cost of $u^{\mathrm{f}} + c^{\mathrm{f}}$. However, when the procurement cost from the flexible supplier is not cheap, the separation phenomenon no longer exists. For instance, Equation (11) shows that the optimal capacity reservation level with a recourse option is a function of $\bar{\theta}^1 \times \bar{\theta}^2$. However, as we will see in the case without recourse (see Theorem 7), the optimal capacity reservation level is the sum of two independent terms: one a function of $\bar{\theta}^1$ and the other a function of $\bar{\theta}^2$.

4.3. *Recourse analysis with offshore unreliable suppliers*

In some situations, a firm may not be able to monitor its unreliable suppliers before placing the orders. For instance, unlike our motivating examples discussed in the Introduction, there might be geographical or other barriers between such suppliers and the firm that prevent the firm from effectively monitoring its unreliable suppliers. Thus, we now assume that the firm first places the orders with unreliable suppliers, observes the delivered quantities, and then places orders with the backup supplier. Specifically, the firm first decides to reserve a capacity of $\bar{Q}^{\mathrm{f}}$ units from the secondary flexible backup supplier and pays $u^{\mathrm{f}} \times \bar{Q}^{\mathrm{f}}$ to do so. Simultaneously, the firm places orders with the dedicated unreliable suppliers (Stage 1). If the corresponding supplier is up, the orders are fully delivered, and the firm pays the full purchasing price to the supplier. Otherwise, nothing is delivered and, therefore, the firm does not pay the purchasing price (i.e., as before, the firm pays only per item delivered). The firm can then use its flexible backup supplier subject to the reserved capacity, $\bar{Q}^{\mathrm{f}}$, by paying the capacity reservation exercise prices (Stage 2). Then, demands are realized and either inventory shortage cost or holding minus the sales revenue accrue.

Let $C_{\mathrm{U,U}}(\bar{Q}^{\mathrm{f}}, q^1, q^2)$, $C_{\mathrm{U,D}}(\bar{Q}^{\mathrm{f}}, q^1)$, $C_{\mathrm{D,U}}(\bar{Q}^{\mathrm{f}}, q^2)$, and $C_{\mathrm{D,D}}(\bar{Q}^{\mathrm{f}})$ (U: up, D: down) denote the minimum expected cost of the firm in the second stage, if both suppliers deliver (i.e., has been up), only the first supplier delivers, only the second supplier delivers, and when none of them delivers, respectively. Analogous to programs (3) to (6), these costs can be computed as follows.

$$C_{\mathrm{U,U}}(\bar{Q}^{\mathrm{f}}, q^1, q^2) = c^1 q^1 + c^2 q^2 + \min_{q_1^{\mathrm{f}}, q_2^{\mathrm{f}} \geq 0 \text{ s.t. } q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \leq \bar{Q}^{\mathrm{f}}} \sum_{j=1}^{2} c_j^{\mathrm{f}} q_j^{\mathrm{f}} + G_1\big(q^1 + q_1^{\mathrm{f}}\big) + G_2\big(q^2 + q_2^{\mathrm{f}}\big), \tag{12}$$

$$C_{\mathrm{U,D}}(\bar{Q}^{\mathrm{f}}, q^1) = c^1 q^1 + \min_{q_1^{\mathrm{f}}, q_2^{\mathrm{f}} \geq 0 \text{ s.t. } q_1^{f} + q_2^{\mathrm{f}} \leq \bar{Q}^{\mathrm{f}}} \sum_{j=1}^{2} c_j^{\mathrm{f}} q_j^{\mathrm{f}} + G_1\big(q^1 + q_1^{\mathrm{f}}\big) + G_2\big(q_2^{\mathrm{f}}\big), \tag{13}$$

$$C_{\mathrm{D,U}}(\bar{Q}^{\mathrm{f}}, q^2) = c^2 q^2 + \min_{q_1^{\mathrm{f}}, q_2^{\mathrm{f}} \geq 0 \text{ s.t. } q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \leq \bar{Q}^{\mathrm{f}}} \sum_{j=1}^{2} c_j^{\mathrm{f}} q_j^{\mathrm{f}} + G_1\big(q_1^{\mathrm{f}}\big) + G_2\big(q^2 + q_2^{\mathrm{f}}\big), \tag{14}$$

$$C_{\mathrm{D,D}}(\bar{Q}^{\mathrm{f}}) = \min_{q_1^{\mathrm{f}}, q_2^{\mathrm{f}} \geq 0 \text{ s.t. } q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \leq \bar{Q}^{\mathrm{f}}} \sum_{j=1}^{2} c_j^{\mathrm{f}} q_j^{\mathrm{f}} + G_1\big(q_1^{\mathrm{f}}\big) + G_2\big(q_2^{\mathrm{f}}\big). \tag{15}$$

Next, if $C_{\mathrm{Stage\,2}}(\bar{Q}^{\mathrm{f}}, q^1, q^2)$ denotes the optimal expected cost of Stage 2 as is perceived by the firm at the beginning of Stage 1, we have:

$$\begin{aligned} C_{\mathrm{Stage\,2}}(\bar{Q}^{\mathrm{f}}, q^1, q^2) = {} & \theta^1 \theta^2 C_{U,U}(\bar{Q}^{\mathrm{f}}, q^1, q^2) \\ & + \theta^1 (1-\theta^2) C_{\mathrm{U,D}}(\bar{Q}^{\mathrm{f}}, q^1) \\ & + (1-\theta^1)\theta^2 C_{\mathrm{D,U}}(\bar{Q}^{\mathrm{f}}, q^2) \\ & + (1-\theta^1)(1-\theta^2) C_{\mathrm{D,D}}(\bar{Q}^{\mathrm{f}}). \end{aligned} \tag{16}$$

Then, the firm can determine $\bar{Q}^{\mathrm{f}*}$ as well as q^{1*}, q^{2*} at the beginning of Stage 1 via

$$\min_{q^1, q^2, \bar{Q}^{\mathrm{f}} \geq 0} \quad u^{\mathrm{f}}\, \bar{Q}^{\mathrm{f}} + C_{\mathrm{Stage\,2}}(\bar{Q}^{\mathrm{f}}, q^1, q^2). \tag{17}$$

Notice that the solution to Equation (15) is already given in Theorem 3, although the value of $\bar{Q}^{\mathrm{f}}$ may differ. Hence, we now need to solve programs (12) to (14).

Theorem 5. (*Both suppliers up*). *For* $j = 1, 2$, *let*

$$\alpha_j = \left[F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j} \right) - q^j \right]^+ \text{ and}$$

$$\beta_j = \left[F_j^{-1}\left(\left[\frac{p_j + r_j - c_j^{\mathrm{f}} - t}{p_j + r_j + h_j} \right]^+ \right) - q^j \right]^+,$$

where t *is the solution to* $\sum_{j=1}^{2} \beta_j = \bar{Q}^{\mathrm{f}}$. *If* $\bar{Q}^{\mathrm{f}} \in [\sum_{j=1}^{2} \alpha_j\,,\, \infty)$, *then* $q_j^{\mathrm{f}*} = \alpha_j$. *Otherwise*, $q_j^{\mathrm{f}*} = \beta_j$.

The above theorem states that, if the reserved capacity from the backup supplier is enough, the firm will set the order-up-to level of product j equal to $F_j^{-1}((p_j + r_j - c_j^{\mathrm{f}})/(p_j + r_j + h_j))$; otherwise, to set the order-up-to level, the firm rations the available backup capacity between the two products using parameter t. Here, t can be thought of as an additional fictitious capacity reservation exercise cost that rations the limited capacity. In each of these cases, and for both products, if the delivered order from the dedicated supplier is more than the order-up-to level, the firm will not use the backup capacity. Otherwise, the firm brings its inventory level to the order-up-to level by reordering the rest of its requirement from the backup supplier and paying the capacity reservation exercise cost.

Next, we solve programs (13) and (14) to analyze the case where exactly one of the dedicated suppliers fails to deliver.

Theorem 6. (*One supplier up*). *Let* $m \in \{1, 2\}$ *denote the dedicated supplier that delivers, and* $n = 3 - m$ *be the disrupted supplier. Let*

$$\alpha_m = \left[F_m^{-1}\left(\frac{p_m + r_m - c_m^{\mathrm{f}}}{p_m + r_m + h_m} \right) - q^m \right]^+,$$

$$\alpha_n = F_n^{-1}\left(\frac{p_n + r_n - c_n^{\mathrm{f}}}{p_n + r_n + h_n} \right),$$

$$\beta_m = \left[F_m^{-1}\left(\left[\frac{p_m + r_m - c_m^{\mathrm{f}} - t}{p_m + r_m + h_m} \right]^+ \right) - q^m \right]^+,$$

and

$$\beta_n = F_n^{-1}\left(\left[\frac{p_n + r_n - c_n^{\mathrm{f}} - t}{p_n + r_n + h_n} \right]^+ \right),$$

where t *is the solution to* $\sum_{j=m,n} \beta_j = \bar{Q}^{\mathrm{f}}$. *If* $\bar{Q}^{\mathrm{f}} \in [\sum_{j=m,n} \alpha_j\,,\, \infty)$, *then* $q_j^{\mathrm{f}*} = \alpha_j$ *for* $j \in \{m, n\}$. *Otherwise*, $q_j^{\mathrm{f}*} = \beta_j$.

The above result states that the firm will set the order-up-to levels as if both unreliable suppliers have delivered. However, unlike the case where both suppliers deliver, the firm can only reach the order-up-to level for product n via the flexible backup supplier. Hence, for instance, if the reserved capacity is enough, the firm will always order a positive amount from the backup supplier for product n. When the reserved backup capacity is not enough, the firm will ration the available limited backup capacity, considering the amount delivered for product m.

Since the solution to program (15) is already given in Theorem 3, we have completely solved programs (12) to (15) and, hence, $C_{\text{Stage 2}}(\bar{Q}^{\text{f}}, q^1, q^2)$ is completely computed. It is then straightforward to solve program (17) to characterize the firm's behavior in the first stage. Furthermore, it should be clear that because of the early orders placed with the unreliable suppliers, the optimal cost in program (17) provides an upper bound for the optimal cost in the previous section.

5. Benchmark analyses: no recourse

To generate insights into the value of the recourse option for firms (Questions 2 and 3) and provide some benchmark analyses for Section 4, we now consider the much simpler case where recourse is not allowed. We present the main results here and provide further details about this setting in Appendix C.

Theorem 7. *Without the recourse option, the perceived optimal ordering and contracting decisions for the firm with* $0 \leq \theta^j < 1$ $(j = 1, 2)$ *are*

$$q_j^{\text{f}*} = F_j^{-1}\left(\left[\frac{p_j + r_j - \left(u_j^{\text{f}} - \theta^j c^j\right)/(1 - \theta^j)}{p_j + r_j + h_j}\right]^+\right) \quad (j = 1, 2), \quad (18)$$

$$q^{j*} = F_j^{-1}\left(\frac{p_j + r_j - c^j}{p_j + r_j + h_j}\right) - F_j^{-1}\left(\left[\frac{p_j + r_j - \left(u_j^{\text{f}} - \theta^j c^j\right)/(1 - \theta^j)}{p_j + r_j + h_j}\right]^+\right) \quad (j = 1, 2), \quad (19)$$

$$\bar{Q}^{\text{f}^*} = \sum_{j=1}^{2} q_j^{\text{f}*} = \sum_{j=1}^{2} F_j^{-1}\left(\left[\frac{p_j + r_j - (u_j^{\text{f}} - \theta^j c^j)/(1 - \theta^j)}{p_j + r_j + h_j}\right]^+\right). \quad (20)$$

Theorem 7 shows in Equation (20) the amount of capacity that the firm reserves, where $u_j^{\text{f}} = u^{\text{f}} + c_j^{\text{f}}$. For each of the products, the firm will order in total (from both suppliers of that product) the same amount that it would order if it had a single, reliable, dedicated supplier with linear ordering cost c^j. However, its *perceived* optimal ordering quantity from the flexible supplier is modified to include its unreliability beliefs about dedicated suppliers. It will then procure the rest of its requirements from the unreliable dedicated supplier. Theorem 7 also proves a separability phenomenon in capacity reservation for the two products. As the reader may have expected, the flexible backup supplier in this benchmark setting has capacity reserved at the same levels as if there are two separate backup suppliers. This is intuitive because the joint backup capacity is not decided *a priori* in this section but is part of the optimization. However, as we observed in Section 4, this separability disappears where recourse is allowed.

5.1. *Benchmark setting: the value of the secondary flexible backup supplier*

We denote the true (and not the perceived) value of the flexible backup supplier for the firm by V^{f} and define it as:

$$V^{\text{f}} = C_{\text{T}}(q'^{1*}, q'^{2*}, 0, 0, 0) - C_{\text{T}}\left(q^{1*}, q^{2*}, q_1^{\text{f}*}, q_2^{\text{f}*}, \bar{Q}^{\text{f}*}\right), \quad (21)$$

where $C_{\text{T}}(\cdot)$ (as is defined in Equation (A67) in Appendix C) denotes the true expected cost of the firm under its perceived optimal decisions, and q'^{j*} represents the firm's perceived optimal ordering quantity to dedicated supplier j in the absence of the flexible supplier.

Using the perceived optimal ordering and contracting levels presented in Theorem 7, we now derive the true value of the flexible backup supplier for the firm in the following lemma.

Lemma 1. *The true value of the flexible backup supplier for the firm under the capacity reservation contract is*

$$V^{\text{f}} = \sum_{j=1}^{2} \left[\pi_0^j \left(p_j E(D_j) - G_j\left(q_j^{\text{f}*}\right)\right) - \left(u_j^{\text{f}} - \left(1 - \pi_0^j\right)c^j\right)q_j^{\text{f}*}\right], \quad (22)$$

where $G_j(\cdot)$ *and* $q_j^{\text{f}*}$ *are defined in Equations* (2) *and* (18), *respectively.*

Now that we have a measure for the value of the flexible backup supplier, we can answer an interesting question:

Question 6: If a firm perceives the capacity reservation contract with a flexible backup supplier to be valuable (and hence will wish to form a contract), will such a contract be also *truly* valuable or not (and *vice versa*)?

Theorem 8.

(i) *The firm perceives the capacity reservation contract with the flexible backup supplier to be valuable, if and only if its reliability belief vector* $\mathbf{\Theta} = (\theta^1, \theta^2)$ *satisfies*:

$$\exists j \in \{1, 2\} : \theta^j < \frac{p_j + r_j - u_j^{\text{f}}}{p_j + r_j - c^j}. \quad (23)$$

(ii) *The capacity reservation contract with the flexible backup supplier is* <u>*not*</u> *truly valuable for the firm if for both* $j = 1, 2$:

$$\max\left\{\theta^j, 1 - \pi_0^j\right\} \geq \frac{p_j + r_j - u_j^{\text{f}}}{p_j + r_j - c^j}. \quad (24)$$

From Theorem 8, we observe that if $(p_j + r_j - u_j^{\text{f}})/(p_j + r_j - c^j) \leq \theta^j$ for both $j = 1, 2$, the firm is *lucky* that its belief is true (regardless of whether it is underestimating or overestimating): contracting with the flexible backup supplier is neither perceived to be valuable nor is truly valuable for the firm. Also, a firm that overestimates the reliabilities of both of its dedicated suppliers (i.e., $\epsilon^j > 0$ for both $j = 1, 2$) perceives the flexible backup supplier to be valuable if, and only if, it is truly valuable (for the only if part, follow the proof of Theorem 2). In fact, we have the following observation:

Observation 1. Overestimating reliabilities does not have the danger of mismatching perception and reality regarding the decision of whether or not to reserve some flexible backup capacity.

While the above observation presents a nice property of overestimating, it does not mean that it is more profitable to overestimate the reliabilities. Indeed, we can rigorously prove the following theorem that verifies the intuitive notion that firms with a more accurate reliability belief (achievable through monitoring unreliable suppliers) can benefit more from contracting with a flexible backup supplier in terms of actual cost reduction.

Proposition 4. *The true value of the flexible backup supplier based on the firm's errors in its reliability belief, denoted by* $V^{\mathrm{f}}(\epsilon^1, \epsilon^2)$, *is non-increasing in the degree of disruption risk perception error. That is, for* $j \in \{1, 2\}$ *if* $\epsilon^j > 0$, *let* $0 \le \delta^j < \epsilon^j$ *and if* $\epsilon^j < 0$, *let* $\epsilon^j < \delta^j \le 0$; *then* $V^{\mathrm{f}}(\epsilon^1, \epsilon^2) \le V^{\mathrm{f}}(\delta^1, \delta^2)$.

The (numerical) Study A1 of Appendix C generates more insights into the value of the flexible backup supplier. In particular, the following observation from this study is of interest.

Observation 2. Even with large errors in its reliability perception of the primary suppliers, a firm with sufficiently high profit margin can greatly benefit from reserving some flexible backup capacity (despite the fact that the quality of information is poor).

5.2. *Benchmark setting: the value of disruption risk information*

As the previous section suggested, a firm gains additional benefit if it obtains disruption risk information and removes the uncertainty about the disruption risks of its suppliers. For instance, in the example of Boeing's supply chain discussed in Section 1, monitoring the production problems of AIT in 2006 could help Boeing to protect against the disruption.

Clearly, obtaining disruption risk information is costly (e.g., the cost of establishing a threat level advisory system, providing suppliers with incentives to collaboratively share their related private information, or placing some of the firm's employees at the supplier's site). Hence, there is a trade-off between the cost of obtaining such information and the savings due to better contracting and ordering decisions. To examine this trade-off, we denote by V^{i^j} the value of obtaining perfect information on dedicated supplier j given that this supplier is in state $i \in \{0, +\}$ and define it as

$$V^{i^j} = C_{\mathrm{T}}\left(q^{1*}, q^{2*}, q_1^{\mathrm{f}*}, q_2^{\mathrm{f}*}, \bar{Q}^{\mathrm{f}*}\right) - C_{\mathrm{T}}\left(q^{1\#}, q^{2\#}, q_1^{\mathrm{f}\#}, q_2^{\mathrm{f}\#}, \bar{Q}^{\mathrm{f}\#}|i^j\right), \tag{25}$$

where superscript # on decision variables describes that they are the firm's perceived optimal decisions with respect to the new information (i.e., i^j), and $C_{\mathrm{T}}(\cdot|i^j)$ is the true expected cost of the firm under such decisions (given that dedicated supplier j is in state i^j). The value of the information on the disruption risk of unreliable supplier j ($j = 1, 2$) can then be computed by

$$\begin{aligned} VI^j &= P(\mathbb{1}_{(i^j=+)} = 1)\, V^{+^j} + P(\mathbb{1}_{(i^j=0)} = 1)\, V^{0^j} \\ &= \left(1 - \pi_0^j\right) V^{+^j} + \pi_0^j\, V^{0^j}, \end{aligned} \tag{26}$$

where we have used $P(\mathbb{1}_{(i^j=0)} = 1) = \pi_0^j$. (Recall that $\mathbf{\Pi} = (1 - \pi_0^1, 1 - \pi_0^2)$ is the vector of *true* reliabilities.) It is noteworthy that the value of information on dedicated supplier j (VI^j) defined in Equation (26) also represents an upper bound for the amount of money that a risk-neutral firm should be willing to pay to obtain information about the disruption risk of dedicated supplier j.

To compute the total value of information, we can define the aggregate value of information for the firm as $VI = \sum_{j=1}^{2} VI^j$, representing the savings that the firm can obtain in its true expected costs by moving from a no-information situation to a full/perfect information one, as computed in the following lemma.

Lemma 2. *The values of the information on the disruption risk of the dedicated supplier* j ($j = 1, 2$) *under the capacity reservation contract for the firm are*

$$\begin{aligned} V^{+^j} &= \left(u_j^{\mathrm{f}} - \left(1 - \pi_0^j\right) c^j\right) q_j^{\mathrm{f}*} + \pi_0^j \left[G_j\left(q_j^{\mathrm{f}*}\right) - G_j\left(q_j^{\mathrm{f}*} + q^{j*}\right) - c^j\left(q_j^{\mathrm{f}*} + q^{j*}\right)\right], \end{aligned} \tag{27}$$

$$\begin{aligned} V^{0^j} &= (1 - \pi_0^j)\left[c^j q^{j*} + G_j\left(q^{j*} + q_j^{\mathrm{f}*}\right) - G_j\left(F_j^{-1}\left(\frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j + h_j}\right)\right)\right] \\ &\quad + \pi_0^j\left[G_j\left(q_j^{\mathrm{f}*}\right) - G_j\left(F_j^{-1}\left(\frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j + h_j}\right)\right)\right] \\ &\quad - u_j^{\mathrm{f}}\left(F_j^{-1}\left(\frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j + h_j}\right) - q_j^{\mathrm{f}*}\right), \end{aligned} \tag{28}$$

$$\begin{aligned} VI^j &= \left(u_j^{\mathrm{f}} - \left(1 - \pi_0^j\right) c^j\right) q_j^{\mathrm{f}*} \\ &\quad + \pi_0^j\left[G_j\left(q_j^{\mathrm{f}*}\right) - G_j\left(F_j^{-1}\left(\frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j + h_j}\right)\right) - u_j^{\mathrm{f}} F_j^{-1}\left(\frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j + h_j}\right)\right], \end{aligned} \tag{29}$$

where $G_j(\cdot)$, q_j^{f*}, and q^{j*} are defined in Equations (2), (18), *and* (19) *respectively.*

Now that we have a measure for the value of disruption risk information, we can answer two interesting questions:

Question 7: When is obtaining disruption risk information a better risk mitigation mechanism than contracting with a flexible backup supplier?
Question 8: For which firms should obtaining disruption risk information be more appealing?

We first provide insight into Question 7.

Proposition 5. *There exist thresholds $\hat{\pi}_0^j$ on the true unreliability of the primary suppliers such that $VI \geq V^f$ whenever $\pi_0^j \leq \hat{\pi}_0^j$ ($j \in \{1, 2\}$). That is, when suppliers are* (*truly*) *reliable enough, obtaining information is more valuable than contracting with a flexible backup supplier.*

When primary suppliers are reliable enough, investing in an expensive backup supplier is not advantageous. However, obtaining information is still (relatively) advantageous because it helps the firm to make better ordering decisions.

The following proposition (and later Observation 3) provide answers to Question 8. First, as is intuitively expected, firms that currently do not have an accurate vector of reliability belief can achieve larger savings in their true costs through obtaining risk information. Hence, monitoring suppliers should be more attractive for such firms.

Proposition 6. *The value of information based on the firm's errors in its reliability belief, denoted by $VI(\epsilon^1, \epsilon^2)$, is nondecreasing in the degree of disruption risk perception error. That is, for $j \in \{1, 2\}$ if $\epsilon^j > 0$, let $0 \leq \delta^j < \epsilon^j$ and if $\epsilon^j < 0$, let $\epsilon^j < \delta^j \leq 0$; then $VI(\epsilon^1, \epsilon^2) \geq VI(\delta^1, \delta^2)$. Furthermore, if $|\delta^j| \geq |\epsilon^j|$, then $VI(\epsilon^1, \epsilon^2) \leq VI(\delta^1, \delta^2)$.*

The (numerical) Study 2 of Online Appendix C provides the following observations about disruption risk information.

Observation 3. Disruption risk information is more attractive to firms with lower profit margins.

Observation 4. The sensitivity of the value of information to belief errors is much higher for firms that tend to overestimate reliabilities than those who underestimate.

6. The value of recourse, flexibility, and information

We now use our framework with recourse to gain insights into the value of three disruption mitigation mechanisms: obtaining (i) a recourse option; (ii) flexibility in the backup supply system; and (iii) disruption risk information on primary suppliers.

6.1. *The value of recourse*

To reveal the benefit of recourse and provide insights to Question 3, we now compare the optimal cost of the model with the recourse to the benchmark setting of Section 5 (where the firm cannot observe the state of the unreliable suppliers before ordering).

Study 1: (Recourse). As in Studies A1 and A2 (described in detail in Appendix C), consider a firm that is facing normally distributed demands of $N(5000, 1200^2)$ and $N(3000, 800^2)$, respectively, for products 1 and 2. Table 1 describes the percentage benefit of recourse as well as the percentage reduction in optimal investment in the flexible capacity based on the parameter settings of Table A2 (see Appendix B). To focus on the effect of recourse, we assume that (i) the firm has no error in its risk belief and (ii) c_j^f ($j = 1, 2$) is negligible compared to u^f. By comparing the settings with recourse and without it in Table 1, we gain the following insights into Questions 2 and 3.

Observation 5. Recourse is a strong mitigation technique for firms with a cost reduction that is always positive and averages 37.7% in our study.

Since this reduction in cost is due to making more effective procurement decisions as a consequence of observing disruption states, the above analysis gives quantitative evidence of the value of postponing ordering decisions (to the extent possible) until after monitoring the state of unreliable suppliers. Another interesting observation is the following.

Observation 6. Investment in the flexible backup capacity may be greater or smaller with recourse.

One might think that a firm with recourse would always invest less in the flexible capacity, since later it can benefit from its disruption observation to flexibly utilize the reserved pooled capacity (a *risk pooling effect*). However, this is not true in Settings 3, 4, and 7. For instance, in Setting 4, the firm without recourse does not invest in the flexible capacity (according to Theorem 8). With a recourse option, however, it knows that if in the second stage it observes that (at least) one of the suppliers is down, it has much to gain by channeling the reserved capacity to the appropriate product(s). Hence, it prefers to invest in the secondary supplier to reduce the risk.

6.2. *The value of flexibility*

We now use our analytical framework to provide insights into Question 4.

Study 2: (Flexibility). To capture the value of implementing flexibility in the backup system, we compare two scenarios: (i) one with two dedicated (i.e., inflexible) backup suppliers, where the dedicated backup supplier of product j has a capacity reservation cost of u_j^d; and (ii) one with a

Table 1. Value of recourse and the difference in investment in the flexible backup capacity with and without recourse

	Recourse		*No recourse*			
Setting no.	$\bar{Q}^{f*}$	*Opt. cost*	$\bar{Q}^{f*}$	*Opt. cost*	*Reduction in* $\bar{Q}^{f*}$ (%)	*Value of recourse (%)*
1	3314	−6766.7	6845	−3156.0	51.59	114.41
2	3287	−6673.4	5415	−3563.2	39.30	87.29
3	3214	−7196.0	2571	−4771.7	−25.01	50.81
4	2459	−7495.7	0	−7050.8	$-\infty$	6.31
5	5206	−93662.7	8308	−90893.4	37.34	3.05
6	5095	−26510.3	8495	−25588.9	40.02	3.60
7	3233	−26484.6	2436	−23940.9	−32.72	10.62
8	3520	−24025.0	6432	−19 086.8	45.27	25.87
			Average			37.744

single flexible backup supplier. Notice that, for computational purposes, the first scenario is a special case of our modeling framework, and to analyze it one can simply use the results provided in Section 4 twice (i.e., separately for each product), each time setting the demand for one of the products to zero. To investigate the value of flexibility, we assume that $u^{\mathrm{d}}_j = (1 + \Delta)c^j$, where Δ represents a "backup premium." Then, to fairly price the capacity of the flexible supplier, we consider u^{f} as a weighted average (based on quantities demanded for each) of u^{d}_1 and u^{d}_2 and set $u^{\mathrm{f}} = [\sum_{j=1,2} E(D_j)\,\bar{\theta}^j\, u^{\mathrm{d}}_j]/[\sum_{j=1,2} E(D_j)\,\bar{\theta}^j]$. The other parameter settings and assumptions are the same as those used in Study 1. The results presented in Table 2 lead to the following two observations.

Observation 7. The (mix) flexibility of a backup supplier is highly beneficial to a firm that is procuring from unreliable primary suppliers, with an average cost reduction of 36.7% in our study. Moreover, the value of implementing flexibility in the backup system increases as the backup premium increases.

Observation 8. The capacity reserved with a single flexible backup supplier is not always less than the total capacity reserved with two dedicated backup suppliers. However, the flexibility results in an average reduction of 23.5% (in our study) in the total backup capacity purchased.

6.3. *The value of disruption risk information*

Finally, we use our analytical framework to gain insights into Question 5 of the introduction to Section 4.

Study 3. (Information). Consider the parameter settings of Study 1 (Table A2 of Appendix B) but assume that the firm's disruption risk belief (θ^1, θ^2) is subject to errors as presented in Table 3. Table 3 presents the value of (perfect) information by comparing the cost of the firm when its decision is based only on its belief (imperfect information) with a scenario where it obtains risk information and decides based on the true risk of its suppliers (perfect information). From Table 3 we see that obtaining information may decrease or increase the firm's investment level in flexible backup capacity, depending on whether the firm has been overestimating or underestimating the risks. Furthermore, we gain the following insight into Question 8.

Observation 9. Obtaining perfect disruption risk information with a recourse option results in an average cost reduction of 4.12%, and it is not a very strong risk mitigation technique compared to obtaining a recourse option and/or implementing flexibility in the backup system.

In other words, once the firm obtains a recourse option to benefit from the flexible backup capacity, the additional benefit of reducing risk belief errors is modest. Additional insights regarding the role of profit margin on the value of disruption information are found in Appendix C.

7. Summary of findings and conclusion

We developed a rigorous quantitative methodology to capture the value of two key supply risk mitigation mechanisms: (i) contracting with a secondary flexible backup supplier and (ii) obtaining disruption risk information through monitoring primary suppliers. We derived analytical measures for the true value of a flexible backup supplier as well as the value of obtaining disruption risk information. These measures determine upper bounds for the amount of money that a risk-neutral firm should be willing to invest to implement either of these strategies in order to increase the reliability and responsiveness of its supply chain.

In both settings with and without a recourse option, we analytically characterized the firm's behavior by explicitly identifying the jointly optimal size of the backup capacity reservation contract and the inventory ordering policy for both products. This characterization was based upon the firm's perception of the primary suppliers' disruption risks.

We observed that investing in a secondary flexible backup capacity can be harmful if the current information about the risk of primary suppliers is not perfect. We showed that monitoring unreliable suppliers to make better

Table 2. The value of flexibility and the reduction in investment in the backup capacity due to flexibility

	Value of flexibility (%) Δ					*Reduction in total capacity reserved (%)* Δ				
Setting no.	*0%*	*5%*	*10%*	*15%*	*Avg. (%)*	*0%*	*5%*	*10%*	*15%*	*Avg. (%)*
1	0.5	16.6	28.0	44.0	22.3	31.6	36.6	16.6	36.0	30.2
2	8.3	16.5	31.6	36.2	23.2	24.9	45.2	16.5	39.6	31.5
3	4.6	13.1	24.9	37.5	20.0	22.3	46.7	13.1	55.8	34.5
4	6.8	16.0	22.3	17.6	15.7	1.6	49.7	16.0	5.1	18.1
5	0.5	1.0	1.6	2.1	1.3	9.3	13.5	1.0	36.4	15.0
6	0.0	3.6	9.3	9.9	5.7	104.9	0.0	3.6	40.3	37.2
7	101.7	103.2	104.9	107.4	104.3	104.0	−54.1	103.2	−48.6	26.1
8	96.7	97.4	104.0	105.8	101.0	40.8	−100.6	97.4	−55.7	−4.5
Avg. (%)	27.4	33.4	40.8	45.1	36.78	42.4	4.6	33.4	13.6	23.52

risk estimates enhances the benefit of purchasing flexible backup capacity. We also identified conditions under which a firm is lucky in the sense that regardless of whether it is overestimating or underestimating the reliabilities, it perceives investing in a flexible backup capacity to be valuable only if it is truly valuable. For instance, we found that overestimating supplier reliabilities does not have the danger of mismatching perception and reality regarding a decision to reserve some flexible backup capacity. Moreover, we showed that contracting with a flexible backup supplier is more beneficial for firms with low perception errors about the reliability of their suppliers than those with high errors. By contrast, disruption risk information is more valuable for firms with higher perception errors. Additionally, we observed that disruption risk information is more attractive to firms with low profit margins than those with high ones. We also showed that when suppliers are (truly) reliable enough, obtaining information is a better risk mitigation technique than contracting with a flexible supplier. We also found that the value of disruption risk information is much more sensitive to the misperception errors for firms who tend to overestimate (rather than underestimate) the reliabilities.

Next, comparing the scenarios with and without the recourse option, our study found that having the recourse option can be regarded as an effective risk mitigation technique for firms with an average cost reduction of 37%. This observation sheds more light on the benefit of monitoring suppliers and provides further evidence that firms with unreliable suppliers should try to postpone (to the extent possible) their ordering decisions until after monitoring the disruption state of their suppliers. We also observed that the amount of investment in the flexible backup capacity may or may not be reduced when a firm obtains a recourse option. Furthermore, we showed that when the perceived reliability of the suppliers is larger than a critical fraction, having a recourse option reduces the optimal investment in the flexible backup capacity.

We investigated the value of implementing flexibility in the backup system: contracting with a single flexible backup supplier rather than two inflexible ones. Our study showed an average cost reduction of 36%, so flexibility can indeed be highly beneficial; furthermore, it becomes more beneficial as the backup premium increases.

We evaluated the benefit of obtaining risk information under the recourse option, but our results suggest that it is not a strong mitigation mechanism. Without recourse, it is potent. We also extended our two-stage analyses to a similar setting with offshore unreliable suppliers, where the availability of unreliable suppliers can only be identified by observing the delivered quantities.

Table 3. The value of disruption risk information under recourse

	Belief error		*Perfect info.*		*Imperfect info.*			
Setting no.	ϵ^1	ϵ^2	$\bar{Q}^{f*}$	*Opt. cost*	$\bar{Q}^{f*}$	*Opt. cost*	*Reduction in* $\bar{Q}^{f*}$ *(%)*	*Value of info. (%)*
1	0.1	0.1	3886	−7028.4	3314	−6956.7	−17.26	1.03
2	−0.1	−0.1	0	−7 861.4	3287	−7334.1	∞	7.19
3	0.1	−0.1	3965	−6136.1	3214	−6003.4	−23.40	2.21
4	0.2	0.15	3924	−6 581.8	2459	−5,701.4	−59.58	15.44
5	0.15	0.2	7463	−91 331.9	5206	−90 117.2	−43.37	1.35
6	−0.1	0.2	3755	−26660.9	5095	−26548.1	26.29	0.42
7	0.2	0.2	4664	−26504.6	3233	−25654.2	−44.29	3.31
8	0.15	0.2	4486	−24457.9	3520	−23977.1	−27.46	2.01
				Average				4.121

We leave it to future research to investigate the effects of dynamic changes in the reliability of the suppliers on the results provided in this article. Future research may also investigate the multi-period trade-offs in carrying inventory over time and dynamically monitoring suppliers to hedge against such dynamically changing risks. Another possible direction for future research is to investigate the effect of correlation in disruption risk across different suppliers and/or correlation in demand across different products on the results provided in this study.

Acknowledgments

The work of both authors was partially supported by NSF grants DMI-0542063 and CMMI-1068636. The authors thank Wallace Hopp, Volodymyr Babich, and the anonymous AE and referees for their valuable suggestions on the results provided in this study.

References

Anupindi, R. and Akella, R. (1993) Diversification under supply uncertainty. *Management Science*, **39**(8), 944–963.

Arreola-Risa, A. and De Croix, G.A. (1998) Inventory management under random disruptions and partial back-orders. *Naval Research Logistics*, **45**, 687–703.

Babich, V. (2006) Vulnerable options in supply chains: effects of supplier competition. *Naval Research Logistics*, **53**(7), 656–673.

Babich, V., Burnetas, A.N. and Ritchken, P.H. (2007) Competition and diversification effects in supply chains with supplier default risk. *Manufacturing Service & Operations Management*, **9**(2), 123–146 .

Bielecki, T. and Kumar, P.R. (1988) Optimality of zero-inventory policies for unreliable manufacturing systems. *Operations Research*, **36**, 532–541.

Chopra, S., Reinhardt, G. and Mohan, U. (2007) The importance of decoupling recurrent and disruption risks in a supply chain. *Naval Research Logistics*, **54**, 544–555.

Dada, M., Petruzzi, N. and Schwarz, L. (2007) A newsvendor's procurement problem when suppliers are unreliable. *Manufacturing Service & Operations Management*, **9**, 9–32.

Eppen, G.D. and Iyer, A.V. (1997) Backup agreements in fashion buying: the value of upstream flexibility. *Management Science*, **43**, 1469–1484.

Federgruen, A. and Yang, N. (2009) Optimal supply diversification under general supply risks. *Operations Research*, **57**, 1451–1468.

Fine, C.H. and Freund, R.M. (1990) Optimal investment in product flexible manufacturing capacity. *Management Science*, **36**, 449–466.

Fisher, M. and Raman, A. (1996) Reducing the cost of demand uncertainty through accurate response to early sales. *Operations Research*, **44**(1), 87–99.

Gerwin, D. (1993) Manufacturing flexibility: a strategic perspective. *Management Science*, **39**, 395–410.

Graves, S.G. and Tomlin, B. (2003) Process flexibility in supply chains. *Management Science*, **49**, 907–919.

Greising, D. and Johnsson, J. (2007) Behind Boeing 787 delays: problems at one of the smallest suppliers in Dreamliner program causing ripple effect. *Chicago Tribune*, December 8.

Gupta, D. (1996) The (Q, r) inventory system with an unreliable supplier. *INFOR*, **34**, 59–76.

Gürler, U. and Parlar, M. (1997) An inventory problem with two randomly available suppliers. *Operations Research*, **45**, 904–918.

Hendricks, K.B. and Singhal, V.R. (2005) An empirical analysis of the effect of supply chain disruption on long-run stock price performance and equity risk of the firm. *Production and Operations Management*, **14**(1), 35–52.

Henig, M., Gerchak, Y., Ernst, R. and Pyke, D.F. (1997) An inventory model embedded in designing a supply contract. *Management Science*, **43**, 184–189.

Iravani, S.M., Kolfal, B. and Van Oyen, M.P. (2011). Capability flexibility: a decision support methodology for production and service systems with flexible resources. *IIE Transactions*, **43**, 363–382.

Iravani, S.M., Van Oyen, M.P. and Sims, K.T. (2005) Structural flexibility: A new perspective on the design of manufacturing and service operations. *Management Science*, **51**, 151–166.

Jordan, W.C. and Graves, S.C. (1995) Principles on the benefits of manufacturing flexibility. *Management Science*, **41**, 577–594.

Kleindorfer, P.R., Belke, J.C., Elliot, M.R., Lee, K., Lowe, R.A. and Feldman, H. (2003) Accident epidemiology and the U.S. chemical industry: accident history and worst-case data from RMP*Info. *Risk Analysis*, **23**(5), 865–881.

Kleindorfer, P.R. and Saad, G.H. (2005) Managing disruption risk in supply chain. *Production and Operations Management*, **14**(1), 53–58.

Kouvelis, P. and Vairaktarakis, G. (1998) Flowshops with processing flexibility across production stages. *IIE Transactions*, **30**, 735–746.

Lee, H.L., Padmanabhan, V. and Whang, S. (1997) Information distortion in a supply chain: the bullwhip effect. *Management Science*, **43**, 546–558.

Li, J., Chand, S., Dada, M. and Mehta, S. (2009) Managing inventory over a short season: models with two procurement opportunities *Manufacturing & Service Operations Management*, **11**(1), 174–184.

Meyer, R.R., Rothkopf, M.H. and Smith, S.A. (1979). Reliability and inventory in a production-storage system. *Management Science*, **25**, 799–807.

Milner, J.M. and Kouvelis, P. (2002) On the complementary value of accurate demand information and production and supplier flexibility. *Manufacturing & Service Operations Management*, **4**(2), 99–113.

Moinzadeh, K. and Aggarwal, P. (1997) Analysis of a production/inventory system subject to random disruptions. *Management Science*, **43**, 1577–1588.

Parlar, M. (1997) Continuous review inventory problem with random supply interruptions. *European Journal of Operational Research*, **99**, 366–385.

Parlar, M. and Berkin, D. (1991) Future supply uncertainty in EOQ models. *Naval Research Logistics*, **38**, 107–121.

Parlar, M. and Perry, D. (1995). Analysis of a (Q, r, T) inventory policy with deterministic and random yields when future demand is uncertain. *European Journal of Operational Research*, **84**, 431–443.

Parlar, M. and Perry, D. (1996) Inventory models of future supply uncertainty with single and multiple suppliers. *Naval Research Logistics*, **43**, 191–210.

Reitman, V. (1997) Toyota Motor shows its mettle after fire destroys parts plant. *Wall Street Journal*, May 8.

Saghafian, S., Van Oyen, M.P. and Kolfal, B. (2011) The "W" network and the dynamic control of unreliable flexible servers. *IIE Transactions*, **43**(12), 893–907.

Serel, D.A., Dada, M. and Moskowitz, H. (2001) Sourcing decision with capacity reservation contract. *European Journal of Operational Research*, **131**, 635–648.

Sethi, A.K. and Sethi, S.P. (1990) Flexibility in manufacturing systems: a survey. *International Journal of Flexible Manufacturing System*, **2**, 289–328.

Sheffi, Y. (2007) *The Resilient Enterprise: Overcoming Vulnerability for Competitive Advantage*, The MIT Press, Cambridge, MA.

Song, J.-S. and Zipkin, P.H. (1996) Inventory control with information about supply conditions. *Management Science*, **42**, 1411–1419.

Suarez, F.F., Cusumano, M.A. and Fine, C.H. (1995) An empirical study of flexibility in manufacturing. *Sloan Management Review*, **37**, 25–32.

Tomlin, B. (2006) On the value of mitigation and contingency strategies for managing supply-chain disruption risks. *Management Science*, **52**(5), 639–657.
Tomlin, B. (2009) The impact of supply-learning on a firms sourcing strategy and inventory investment when suppliers are unreliable. *Manufacturing & Service Operations Management*, **11**, 192–209.
Tomlin, B. and Snyder, L. (2006). On the value of a threat advisory system for managing supply chain disruptions. Working paper, Kenan-Flager Business School, University of North Carolina, Chapel Hill, NC.
Tomlin, B. and Wang, Y. (2005) On the value of mix flexibility and dual sourcing in unreliable newsvendor networks. *Manufacturing & Service Operations Management*, **7**, 37–57.
Van Mieghem, J.A. (1998) Investment strategies for flexible resources. *Management Science*, **44**, 1071–1078.
Wang, Y., Gilland, W. and Tomlin, B. (2010) Mitigating supply risk: dual sourcing or process improvement? *Manufacturing & Service Operations Management*, **12**, 489–510.
Yang, Z., Aydin, G., Babich, V. and Beil, D. (2009) Supply disruptions, asymmetric information, and a backup production option. *Management Science*, **55**, 192–209.
Yano, C.A. and Lee, H.L. (1995) Lot sizing with random yield: a review. *Operations Research*, **43**, 311–334.

Appendices

Appendix A: Proofs

Proof of Proposition 1. Note that the KKT conditions are sufficient and necessary for characterizing the optimal solution (i.e., the optimal ordering policy). Let μ, λ^{f}, and λ denote the Lagrangian multipliers for constraints $q^{\mathrm{f}} \le \bar{Q}^{\mathrm{f}}$, $q^{\mathrm{f}} \ge 0$, and $q \ge 0$, respectively. First consider the case where the unreliable supplier is observed to be up. Using the Leibniz rule to get the derivative of the objective function $c^{\mathrm{f}}q^{\mathrm{f}} + cq + G(q+q^{\mathrm{f}})$, the KKT conditions can be written as

$$\begin{aligned}
&q^{\mathrm{f}} \le \bar{Q}^{\mathrm{f}},\\
&c + (h+p+r)F(q+q^{\mathrm{f}}) - p - r = \lambda,\\
&c^{\mathrm{f}} + (h+p+r)F(q+q^{\mathrm{f}}) - p - r = \lambda^{\mathrm{f}} - \mu,\\
&\mu(q^{\mathrm{f}} - \bar{Q}^{\mathrm{f}}) = 0,\\
&\lambda q = 0,\\
&\lambda^{\mathrm{f}} q^{\mathrm{f}} = 0,\\
&q, q^{\mathrm{f}}, \mu, \lambda, \lambda^{\mathrm{f}} \ge 0.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&q^{\mathrm{f}} \le \bar{Q}^{\mathrm{f}},\\
&c - \lambda = \mu - \lambda^{\mathrm{f}} + c^{\mathrm{f}},\\
&q + q^{\mathrm{f}} = F^{-1}\left(\frac{p+r-(c-\lambda)}{p+r+h}\right),\\
&\mu(q^{\mathrm{f}} - \bar{Q}^{\mathrm{f}}) = 0,\\
&\lambda q = 0,\\
&\lambda^{\mathrm{f}} q^{\mathrm{f}} = 0\\
&q, q^{\mathrm{f}}, \mu, \lambda, \lambda^{\mathrm{f}} \ge 0.
\end{aligned}$$

Similarly, when the unreliable supplier is down, the KKT conditions are

$$\begin{aligned}
&q^{\mathrm{f}} \le \bar{Q}^{\mathrm{f}},\\
&q^{\mathrm{f}} = F^{-1}\left(\frac{p+r-(c^{\mathrm{f}}+\mu-\lambda^{\mathrm{f}})}{p+r+h}\right),\\
&\mu(q^{\mathrm{f}} - \bar{Q}^{\mathrm{f}}) = 0,\\
&\lambda^{\mathrm{f}} q^{\mathrm{f}} = 0,\\
&q^{\mathrm{f}}, \mu, \lambda^{\mathrm{f}} \ge 0.
\end{aligned}$$

To prove Part 1, set $q^{\mathrm{f}*} = \mu = \lambda = 0$, $\lambda^{\mathrm{f}} = c^{\mathrm{f}} - c$, and

$$q^* = F^{-1}\left(\frac{p+r-c}{p+r+h}\right),$$

and observe that the KKT conditions are satisfied. To prove Part 2, if

$$\bar{Q}^{\mathrm{f}} \ge F^{-1}\left(\frac{p+r-c^{\mathrm{f}}}{p+r+h}\right),$$

then set

$$q^{\mathrm{f}*} = F^{-1}\left(\frac{p+r-c^{\mathrm{f}}}{p+r+h}\right), \quad \mu = \lambda^{\mathrm{f}} = 0,$$

and observe that KKT conditions are satisfied. Otherwise, choose μ such that

$$\bar{Q}^{\mathrm{f}} = F^{-1}\left(\frac{p+r-(c^{\mathrm{f}}+\mu)}{p+r+h}\right).$$

Then, observe that setting $q^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}}$ and $\lambda^{\mathrm{f}} = 0$ satisfies the KKT conditions. For Part 3 (i), set $\mu = \lambda^{\mathrm{f}} = 0$, $\lambda = c - c^{\mathrm{f}}$, $q^* = 0$, and

$$q^{\mathrm{f}*} = F^{-1}\left(\frac{p+r-c^{\mathrm{f}}}{p+r+h}\right).$$

For Part 3 (ii), choose μ such that

$$\bar{Q}^{\mathrm{f}} = F^{-1}\left(\frac{p+r-(c^{\mathrm{f}}+\mu)}{p+r+h}\right)$$

and set $\lambda^{\mathrm{f}} = 0$, $\lambda = c - c^{\mathrm{f}} + \mu$, $q^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}}$, and $q^* = 0$. For Part 3 (iii), set $\mu = c - c^{\mathrm{f}}$, $\lambda^{\mathrm{f}} = \lambda = 0$, $q^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}}$, and

$$q^* = F^{-1}\left(\frac{p+r-c}{p+r+h}\right) - \bar{Q}^{\mathrm{f}}.$$

Similarly, to prove Part 4 (i), set $\mu = \lambda^{\mathrm{f}} = 0$ and

$$q^{\mathrm{f}*} = F^{-1}\left(\frac{p+r-c^{\mathrm{f}}}{p+r+h}\right),$$

and to prove Part 4 (ii), choose μ such that

$$\bar{Q}^{\mathrm{f}} = F^{-1}\left(\frac{p+r-(c^{\mathrm{f}}+\mu)}{p+r+h}\right)$$

and set $\lambda^{\mathrm{f}} = 0$ and $q^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}}$. ■

Proof of Proposition 2. Recall that the optimal capacity reservation level, similar to Equation (8), is $q^{\bar{f}*} = \arg\min_{\bar{Q}^f \geq 0} u^f \bar{Q}^f + \theta C_U(\bar{Q}^f) + (1-\theta) C_D(\bar{Q}^f)$, where $C_U(\cdot)$ ($C_D(\cdot)$) denotes the cost of Stage 2 if the unreliable supplier is observed to be up (down). For the ease of notation, let

$$l = F^{-1}\left(\frac{p+r-c^f}{p+r+h}\right) \quad \text{and } l' = F^{-1}\left(\frac{p+r-c}{p+r+h}\right).$$

When $c^f > c$, from Proposition 1 Part 1, $C_U(\cdot) = a$ for some constant a. Moreover, from Proposition 1 Part 2, $C_D(\bar{Q}^f) = c^f(l \wedge \bar{Q}^f) + G(l \wedge \bar{Q}^f)$, where $x \wedge y = \min\{x, y\}$. Thus, the optimization problem when $c^f > c$ is $\min_{\bar{Q}^f \geq 0} u^f \bar{Q}^f + (1-\theta)[c^f(l \wedge \bar{Q}^f) + G(l \wedge \bar{Q}^f)]$. Note that since the optimizer of $u^f \bar{Q}^f + (1-\theta)[c^f \bar{Q}^f + G(\bar{Q}^f)]$ on $\bar{Q}^f \geq 0$ is

$$F^{-1}\left(\left[\frac{p+r-(u^f/(1-\theta)+c^f)}{p+r+h}\right]^+\right),$$

which is always less than l, the proof of Part 1 is complete. When $c^f \leq c$, we need to consider three cases: (i) $\bar{Q}^f \in [l, \infty)$; (ii) $\bar{Q}^f \in [l', l]$; and (iii) $\bar{Q}^f \in [0, l']$. In case (i), from Proposition 1 Parts 3 (i) and 4 (i), we have $C_U(\bar{Q}^f) = C_D(\bar{Q}^f) = c^f l + G(l)$. Thus, in this case, the optimization problem is $\min_{\bar{Q}^f \geq l} u^f \bar{Q}^f + c^f l + G(l)$, which has the solution $q^{\bar{f}*} = l$. In case (ii), from Proposition 1 Parts 3 (ii) and 4 (ii), $C_U(\bar{Q}^f) = C_D(\bar{Q}^f) = c^f \bar{Q}^f + G(\bar{Q}^f)$. Hence, the optimization problem is $\min_{l' \leq \bar{Q}^f \leq l} u^f \bar{Q}^f + c^f \bar{Q}^f + G(\bar{Q}^f)$. Note that the unconstrained version of this problem has the optimizer

$$F^{-1}\left(\left[\frac{p+r-(u^f+c^f)}{p+r+h}\right]^+\right) \leq l',$$

where the inequality holds since by our modeling assumption $u^f + c^f > c$. Hence, the optimizer in case (ii) is $q^{\bar{f}*} = l'$, since the objective function is convex. In case (iii), from Proposition 1 Parts 3 (iii) and 4 (ii), $C_U(\bar{Q}^f) = c^f \bar{Q}^f + G(l') + c(l' - \bar{Q}^f)$ and $C_D(\bar{Q}^f) = c^f \bar{Q}^f + G(\bar{Q}^f)$. Thus, the optimization problem in this case is equivalent to $\min_{0 \leq \bar{Q}^f \leq l'} (u^f + c^f - \theta c)\bar{Q}^f + (1-\theta)G(\bar{Q}^f)$, which has the optimizer

$$F^{-1}\left(\left[\frac{p+r-(u^f+c^f-\theta c)/(1-\theta)}{p+r+h}\right]^+\right) \leq l',$$

where the inequality holds since by our modeling assumption $u^f + c^f > c$. Next observe that the optimizer in case (i) is a feasible point in case (ii) and also the optimizer of case (ii) is a feasible point in case (iii). Hence, the optimizer of case (iii) gives the global optimal solution to the problem, and the proof is complete. ∎

Proof of Theorem 1. Consider program (3). Notice that it is a convex program with linear constraints. Hence, the KKT conditions are sufficient and necessary. Using the Leibniz rule these conditions are as follows:

$$\begin{aligned}
&q_1^f + q_2^f \leq \bar{Q}^f \\
&c^j + (h_j + p_j + r_j)F_j(q^j + q_j^f) - p_j - r_j = \lambda^j \\
&\qquad\qquad (j = 1, 2), \quad \text{(A1)} \\
&c_j^f + (h_j + p_j + r_j)F_j(q^j + q_j^f) - p_j - r_j = \lambda_j^f - \mu \\
&\qquad\qquad (j = 1, 2), \quad \text{(A2)} \\
&\mu(q_1^f + q_2^f - \bar{Q}^f) = 0 \\
&\lambda^j q^j = 0 \qquad (j = 1, 2), \\
&\lambda_j^f q_j^f = 0 \qquad (j = 1, 2), \\
&q^j, q_j^f, \mu, \lambda^j, \lambda_j^f \geq 0. \qquad (j = 1, 2).
\end{aligned}$$

Conditions (A1) and (A2) result that:

$$\begin{aligned}
&c^j - \lambda^j = \mu - \lambda_j^f + c_j^f \qquad (j = 1, 2), \\
&q^j + q_j^f = F_j^{-1}\left(\frac{p_j + r_j - (c^j - \lambda^j)}{p_j + r_j + h_j}\right). \qquad (j = 1, 2).
\end{aligned}$$

Hence, the KKT conditions can be written as follows:

$$\begin{aligned}
&q_1^f + q_2^f \leq \bar{Q}^f, && \text{(A3)} \\
&c^j - \lambda^j = \mu - \lambda_j^f + c_j^f \quad (j = 1, 2), && \text{(A4)} \\
&q^j + q_j^f = F_j^{-1}\left(\frac{p_j + r_j - (c^j - \lambda^j)}{p_j + r_j + h_j}\right) \quad (j = 1, 2), && \text{(A5)} \\
&\mu(q_1^f + q_2^f - \bar{Q}^f) = 0, && \text{(A6)} \\
&\lambda^j q^j = 0 \quad (j = 1, 2), && \text{(A7)} \\
&\lambda_j^f q_j^f = 0 \quad (j = 1, 2), && \text{(A8)} \\
&q^j, q_j^f, \mu, \lambda^j, \lambda_j^f \geq 0. \quad (j = 1, 2). && \text{(A9)}
\end{aligned}$$

Now, it is sufficient to show that the optimal solution in each part satisfies the above conditions.

1. If

$$\bar{Q}^f \in \left[\sum_{j=k,l} F_j^{-1}\left(\frac{p_j + r_j - c_j^f}{p_j + r_j + h_j}\right), +\infty\right]$$

set

$$\mu = \lambda_j^f = 0, \quad \text{and} \quad \lambda^j = c^j - c_j^f \ (j = 1, 2).$$

Then observe that

$$q_j^{f*} = F_j^{-1}\left(\frac{p_j + r_j - c_j^f}{p_j + r_j + h_j}\right) \quad \text{and} \quad q^{j*} = 0 \ (j = 1, 2)$$

satisfy the KKT conditions.

2. If

$$\bar{Q}^f \in \left[\sum_{j=k,l} F_j^{-1}\left(\frac{p_j + r_j - c_j^f - (c^l - c_l^f)}{p_j + r_j + h_j}\right),\right.$$
$$\left. \times \sum_{j=k,l} F_j^{-1}\left(\frac{p_j + r_j - c_j^f}{p_j + r_j + h_j}\right)\right],$$

set

$$\mu = t, \quad \lambda^k = c^k - c_k^{\mathrm{f}} - \mu, \quad \lambda^l = c^l - c_l^{\mathrm{f}} - \mu, \quad \text{and} \quad \lambda_k^{\mathrm{f}} = \lambda_l^{\mathrm{f}} = 0.$$

Then observe that

$$q_j^{\mathrm{f}*} = F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}} - t}{p_j + r_j + h_j}\right) \quad \text{and}$$
$$q^{j*} = 0 \quad (\text{for both } j = k, l)$$

satisfy the KKT conditions, where $t \in (0, c^l - c_l^{\mathrm{f}}]$ is a solution to:

$$\sum_{j=k,l} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}} - t}{p_j + r_j + h_j}\right) = \bar{Q}^{\mathrm{f}}.$$

3. If

$$\bar{Q}^{\mathrm{f}} \in \left[F_k^{-1}\left(\frac{p_k + r_k - c_k^{\mathrm{f}} - (c^l - c_l^{\mathrm{f}})}{p_k + r_k + h_k}\right), \sum_{j=k,l} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}} - (c^l - c_l^{\mathrm{f}})}{p_j + r_j + h_j}\right)\right],$$

set

$$\mu = c^l - c_l^{\mathrm{f}}, \quad \lambda_l^{\mathrm{f}} = \lambda_k^{\mathrm{f}} = \lambda^l = 0, \quad \text{and} \quad \lambda^k = (c^k - c_k^{\mathrm{f}}) - (c^l - c_l^{\mathrm{f}}).$$

Then observe that

$$q_k^{\mathrm{f}*} = F_k^{-1}\left(\frac{p_k + r_k - c_k^{\mathrm{f}} - (c^l - c_l^{\mathrm{f}})}{p_k + r_k + h_k}\right),$$
$$q_l^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}} - q_k^{\mathrm{f}*}, q^{k*} = 0, \quad \text{and}$$
$$q^{l*} = F_l^{-1}\left(\frac{p_l + r_l - c^l}{p_l + r_l + h_l}\right) - q_l^{\mathrm{f}*}$$

satisfy the KKT conditions.

4. If

$$\bar{Q}^{\mathrm{f}} \in \left[F_k^{-1}\left(\frac{p_k + r_k - c^k}{p_k + r_k + h_k}\right), F_k^{-1}\left(\frac{p_k + r_k - c_k^{\mathrm{f}} - (c^l - c_l^{\mathrm{f}})}{p_k + r_k + h_k}\right)\right],$$

suppose $t \in (c^l - c_l^{\mathrm{f}}, c^k - c_k^{\mathrm{f}}]$ is a solution to

$$\bar{Q}^{\mathrm{f}} = F_k^{-1}\left(\frac{p_k + r_k - c_k^{\mathrm{f}} - t}{p_k + r_k + h_k}\right).$$

Then set $\mu = t, \lambda^k = c^k - c_k^{\mathrm{f}} - \mu, \lambda^l = \lambda_k^{\mathrm{f}} = 0$, and $\lambda_l^{\mathrm{f}} = \mu - (c^l - c_l^{\mathrm{f}})$. Then observe that

$$q_k^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}}, \quad q_l^{\mathrm{f}*} = 0, \quad q^{k*} = 0 \quad \text{and}$$
$$q^{l*} = F_l^{-1}\left(\frac{p_l + r_l - c^l}{p_l + r_l + h_l}\right)$$

satisfy the KKT conditions.

5. If

$$\bar{Q}^{\mathrm{f}} \in \left[0, F_k^{-1}\left(\frac{p_k + r_k - c^k}{p_k + r_k + h_k}\right)\right],$$

set

$$\mu = c^k - c_k^{\mathrm{f}}, \lambda^l = \lambda^k = \lambda_k^{\mathrm{f}} = 0, \quad \text{and} \quad \lambda_l^{\mathrm{f}} = (c^k - c_k^{\mathrm{f}}) - (c^l - c_l^{\mathrm{f}}).$$

Then observe that

$$q_k^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}}, \quad q_l^{\mathrm{f}*} = 0, \quad q^{k*} = F_k^{-1}\left(\frac{p_k + r_k - c^k}{p_k + r_k + h_k}\right) - \bar{Q}^{\mathrm{f}}, \text{ and}$$
$$q^{l*} = F_l^{-1}\left(\frac{p_l + r_l - c^l}{p_l + r_l + h_l}\right)$$

satisfy the KKT conditions.

∎

Proof of Theorem 2. Consider program (4) or (5) depending on whether $m = 1$ or 2 (respectively). Notice that both programs are convex with linear constraints. Hence, the KKT conditions are sufficient and necessary. Using the Leibniz rule and similar to the derivation of the KKT conditions in the proof of Theorem 1 these conditions are as follows (see the proof of Theorem 1 for more details on the derivation of the KKT conditions):

$$q_m^{\mathrm{f}} + q_n^{\mathrm{f}} \le \bar{Q}^{\mathrm{f}},$$
$$c^m + (h_m + p_m + r_m)F_m(q^m + q_m^{\mathrm{f}}) - p_m - r_m = \lambda^m, \tag{A10}$$
$$c_m^{\mathrm{f}} + (h_m + p_m + r_m)F_m(q^m + q_m^{\mathrm{f}}) - p_m - r_m = \lambda_m^{\mathrm{f}} - \mu, \tag{A11}$$
$$c_n^{\mathrm{f}} + (h_n + p_n + r_n)F_n(q_n^{\mathrm{f}}) - p_n - r_n = \lambda_n^{\mathrm{f}} - \mu, \tag{A12}$$
$$\mu(q_m^{\mathrm{f}} + q_n^{\mathrm{f}} - \bar{Q}^{\mathrm{f}}) = 0,$$
$$\lambda^m q^m = 0,$$
$$\lambda_m^{\mathrm{f}} q_m^{\mathrm{f}} = 0,$$
$$\lambda_n^{\mathrm{f}} q_n^{\mathrm{f}} = 0,$$
$$q^m, q_m^{\mathrm{f}}, q_n^{\mathrm{f}}, \mu, \lambda^m, \lambda_m^{\mathrm{f}}, \lambda_n^{\mathrm{f}} \ge 0.$$

Conditions (A10) to (A12) result in

$$c^m - \lambda^m = \mu - \lambda_m^{\mathrm{f}} + c_m^{\mathrm{f}},$$
$$q^m + q_m^{\mathrm{f}} = F_m^{-1}\left(\frac{p_m + r_m - (\mu - \lambda_m^{\mathrm{f}} + c_m^{\mathrm{f}})}{p_m + r_m + h_m}\right),$$
$$q_n^{\mathrm{f}} = F_n^{-1}\left(\frac{p_n + r_n - (\mu - \lambda_n^{\mathrm{f}} + c_n^{\mathrm{f}})}{p_n + r_n + h_n}\right).$$

Hence, the KKT conditions when only supplier $m \in \{1, 2\}$ turns out to be up can be written as

$$q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \leq \bar{Q}^{\mathrm{f}}, \tag{A13}$$
$$c^m - \lambda^m = \mu - \lambda_m^{\mathrm{f}} + c_m^{\mathrm{f}}, \tag{A14}$$
$$q^m + q_m^{\mathrm{f}} = F_m^{-1}\left(\frac{p_m + r_m - \left(\mu - \lambda_m^{\mathrm{f}} + c_m^{\mathrm{f}}\right)}{p_m + r_m + h_m}\right), \tag{A15}$$
$$q_n^{\mathrm{f}} = F_n^{-1}\left(\frac{p_n + r_n - \left(\mu - \lambda_n^{\mathrm{f}} + c_n^{\mathrm{f}}\right)}{p_n + r_n + h_n}\right), \tag{A16}$$
$$\lambda^m q^m = 0, \tag{A17}$$
$$\lambda_m^{\mathrm{f}} q_m^{\mathrm{f}} = 0, \tag{A18}$$
$$\lambda_n^{\mathrm{f}} q_n^{f} = 0, \tag{A19}$$
$$q^m, q_m^{\mathrm{f}}, q_n^{\mathrm{f}}, \mu, \lambda^m, \lambda_m^{f}, \lambda_n^{f} \geq 0. \tag{A20}$$

Now, it is sufficient to show that the optimal solution in each part satisfies the above conditions.

1. If

$$\bar{Q}^{\mathrm{f}} \in \left[\sum_{j=n,m} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right), +\infty\right],$$

set

$$\lambda^m = c^m, \mu = \lambda_m^{\mathrm{f}} = \lambda_n^{\mathrm{f}} = 0$$

Then observe that

$$q_j^{\mathrm{f}*} = F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right) \quad \text{(for both } j = n, m\text{)}$$
$$\text{and } q^{m*} = 0$$

satisfy the KKT conditions.

2. If

$$\bar{Q}^{\mathrm{f}} \in \left[\sum_{j=n,m} F_j^{-1}\left(\left[\frac{p_j + r_j - c_j^{\mathrm{f}} - \left(c^m - c_m^{\mathrm{f}}\right)}{p_j + r_j + h_j}\right)\right]^+\right), \sum_{j=n,m} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right)\Bigg),$$

set

$$\mu = t, \quad \lambda^m = c^m - c_m^{\mathrm{f}} - t \quad \text{and} \quad \lambda_m^{\mathrm{f}} = \lambda_n^{\mathrm{f}} = 0.$$

Then observe that

$$q_j^{\mathrm{f}*} = F_j^{-1}\left(\left[\frac{p_j + r_j - c_j^{\mathrm{f}} - t}{p_j + r_j + h_j}\right]^+\right) \quad \text{(for both } j = n, m\text{)}$$
$$\text{and} \quad q^{m*} = 0$$

satisfy the KKT conditions, where $t \in (0, c^m - c_m^{\mathrm{f}}]$ is a solution to:

$$\sum_{j=n,m} F_j^{-1}\left(\left[\frac{p_j + r_j - c_j^{\mathrm{f}} - t}{p_j + r_j + h_j}\right]^+\right) = \bar{Q}^{\mathrm{f}}.$$

3. If

$$\bar{Q}^{\mathrm{f}} \in \left[F_n^{-1}\left(\left[\frac{p_n + r_n - c_n^{\mathrm{f}} - \left(c^m - c_m^{\mathrm{f}}\right)}{p_n + r_n + h_n}\right]^+\right), \times \sum_{j=n,m} F_j^{-1}\left(\left[\frac{p_j + r_j - c_j^{\mathrm{f}} - \left(c^m - c_m^{\mathrm{f}}\right)}{p_j + r_j + h_j}\right]^+\right)\right),$$

set

$$\mu = c^m - c_m^{\mathrm{f}} \quad \text{and } \lambda_m^{\mathrm{f}} = \lambda_n^{\mathrm{f}} = \lambda^m = 0.$$

Then observe that

$$q_n^{\mathrm{f}*} = F_n^{-1}\left(\left[\frac{p_n + r_n - c_n^{\mathrm{f}} - \left(c^m - c_m^{\mathrm{f}}\right)}{p_n + r_n + h_n}\right]^+\right),$$
$$q_m^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}} - q_n^{\mathrm{f}*}, \quad \text{and}$$
$$q^{m*} = F_m^{-1}\left(\frac{p_m + r_m - c^m}{p_m + r_m + h_m}\right) - q_m^{\mathrm{f}*}$$

satisfy the KKT conditions.

4. If

$$\bar{Q}^{\mathrm{f}} \in \left[0, F_n^{-1}\left(\left[\frac{p_n + r_n - c_n^{\mathrm{f}} - \left(c^m - c_m^{\mathrm{f}}\right)}{p_n + r_n + h_n}\right]^+\right)\right]$$

let $t \in (c^m - c_m^{\mathrm{f}}, \infty)$ be a solution to

$$F_n^{-1}\left(\left[\frac{p_n + r_n - c_n^{\mathrm{f}} - t}{p_n + r_n + h_n}\right]^+\right) = \bar{Q}^{\mathrm{f}}.$$

Set

$$\mu = t, \quad \lambda_m^{\mathrm{f}} = \mu - \left(c^m - c_m^{\mathrm{f}}\right) \quad \text{and } \lambda_n^{\mathrm{f}} = \lambda^m = 0.$$

Then observe that

$$q_n^{\mathrm{f}*} = \bar{Q}^{\mathrm{f}}, q_m^{\mathrm{f}*} = 0, \text{ and } q^{m*} = F_m^{-1}\left(\frac{p_m + r_m - c^m}{p_m + r_m + h_m}\right)$$

satisfy the KKT conditions.

■

Proof of Theorem 3. Consider program (6). Notice that this program is convex with linear constraints. Hence, the KKT conditions are sufficient and necessary. Using the Leibniz rule and similar to the derivation of the KKT conditions in the proof of Theorems 1 and Theorem 2, these conditions are as follows:

$$q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \leq \bar{Q}^{\mathrm{f}},$$
$$c_j^{\mathrm{f}} + (h_j + p_j + r_j) F_j\left(q_j^{\mathrm{f}}\right) - p_j - r_j = \lambda_j^{\mathrm{f}} - \mu \quad (j = 1, 2), \tag{A21}$$
$$\mu\left(q_1^{\mathrm{f}} + q_2^{\mathrm{f}} - \bar{Q}^{\mathrm{f}}\right) = 0,$$
$$\lambda_j^{\mathrm{f}} q_j^{\mathrm{f}} = 0 \quad (j = 1, 2),$$
$$q_j^{\mathrm{f}}, \mu, \lambda_j^{\mathrm{f}} \geq 0. \quad (j = 1, 2).$$

By rewriting condition (A21), KKT conditions are

$$q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \le \bar{Q}^{\mathrm{f}}, \tag{A22}$$

$$q_j^{\mathrm{f}} = F_j^{-1}\left(\frac{p_j + r_j - \left(\mu - \lambda_j^{\mathrm{f}} + c_j^{\mathrm{f}}\right)}{p_j + r_j + h_j}\right) \quad (j = 1, 2), \tag{A23}$$

$$\mu\left(q_1^{\mathrm{f}} + q_2^{\mathrm{f}} - \bar{Q}^{\mathrm{f}}\right) = 0, \tag{A24}$$

$$\lambda_j^{\mathrm{f}} q_j^{\mathrm{f}} = 0 \quad (j = 1, 2), \tag{A25}$$

$$q_j^{\mathrm{f}}, \mu, \lambda_j^{\mathrm{f}} \ge 0. \quad (j = 1, 2). \tag{A26}$$

Now, it is sufficient to show that the optimal solution of each part satisfies the above KKT conditions.

1. If

$$\bar{Q}^{\mathrm{f}} \in \left[\sum_{j=1,2} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right), +\infty\right],$$

set $\mu = \lambda_1^{\mathrm{f}} = \lambda_2^{\mathrm{f}} = 0$.
Then, observe that

$$q_j^{\mathrm{f}*} = F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right) \quad (\text{for both } j = 1, 2)$$

satisfies the KKT conditions.

2. If

$$\bar{Q}^{\mathrm{f}} \in \left[0, \sum_{j=1,2} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right)\right]$$

set $\mu = t$ and $\lambda_1^{\mathrm{f}} = \lambda_2^{\mathrm{f}} = 0$. Then, observe that

$$q_j^{\mathrm{f}*} = F_j^{-1}\left(\left[\frac{p_j + r_j - c_j^{\mathrm{f}} - t}{p_j + r_j + h_j}\right]^+\right) \quad (\text{for both } j = n, m)$$

satisfies the KKT conditions, where $t \in (0, \max_j\{p_j + r_j - c_j^{\mathrm{f}}\}]$ is a solution to

$$\sum_{j=1,2} F_j^{-1}\left(\left[\frac{p_j + r_j - c_j^{\mathrm{f}} - t}{p_j + r_j + h_j}\right]^+\right) = \bar{Q}^{\mathrm{f}}.$$

■

Proof of Proposition 3. Let

$$I_7 = \left[\sum_{j=1,2} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right), \infty\right)$$

and notice that $\bigcup_{k=1}^{7} I_k = [0, \infty)$. Thus, defining

$$\bar{Q}_k^{\mathrm{f}} = \underset{\bar{Q}^{\mathrm{f}} \in I_k}{\arg\min} \quad u^{\mathrm{f}}\, \bar{Q}^{\mathrm{f}} + C_{\mathrm{Stage}\,2}(\bar{Q}^{\mathrm{f}}),$$

we have:

$$\begin{aligned}\bar{Q}^{\mathrm{f}*} &= \underset{\bar{Q}^{\mathrm{f}} \ge 0}{\arg\min} \quad u^{\mathrm{f}}\, \bar{Q}^{\mathrm{f}} + C_{\mathrm{Stage}\,2}(\bar{Q}^{\mathrm{f}}) \\ &= \underset{\bar{Q}^{\mathrm{f}} \in \{\bar{Q}_k^{\mathrm{f}}, k=1,\ldots,7\}}{\arg\min} \quad u^{\mathrm{f}}\, \bar{Q}^{\mathrm{f}} + C_{\mathrm{Stage}\,2}(\bar{Q}^{\mathrm{f}}).\end{aligned} \tag{A27}$$

It remains to identify $\bar{Q}_k^{\mathrm{f}}$ for $k = 1, 2, \ldots, 7$ using Theorems 1, 2 and 3. First, using part (i) of these theorems, notice that on I_7:

$$C_{\mathrm{Stage}\,2}(\bar{Q}^{\mathrm{f}}) = K_1.$$

for some constant K_1. (We use $K, K_1, \ldots, K_8$ to represent constants throughout this proof.) Thus, since $u^{\mathrm{f}} \ge 0$, we have:

$$\begin{aligned}\bar{Q}_7^{\mathrm{f}} &= \underset{\bar{Q}^{\mathrm{f}} \in I_7}{\arg\min} \quad u^{\mathrm{f}}\, \bar{Q}^{\mathrm{f}} + C_{\mathrm{Stage}\,2}(\bar{Q}^{\mathrm{f}}) \\ &= \min_{\bar{Q}^{\mathrm{f}} \in I_7} \bar{Q}^{\mathrm{f}} = \sum_{j=1,2} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right).\end{aligned}$$

Hence, $\bar{Q}_7^{\mathrm{f}} \in I_6$. Thus, from Equation (A27), $\bar{Q}^{\mathrm{f}*} \in \{\bar{Q}_k^{\mathrm{f}}, k = 1, \ldots, 6\}$. Next, on I_6, using part (ii) of Theorems 1, 2, and 3: $C_{\mathrm{U,U}}(\bar{Q}^{\mathrm{f}}) = C_{\mathrm{U,U}}(\bar{Q}^{\mathrm{f}}) = C_{\mathrm{U,D}}(\bar{Q}^{\mathrm{f}}) = C_{\mathrm{D,U}}(\bar{Q}^{\mathrm{f}}) = C_{\mathrm{D,D}}(\bar{Q}^{\mathrm{f}}) = \Gamma(\bar{Q}^{\mathrm{f}})$. (Notice that although Theorems 1, 2, and 3 are presented based on open end intervals, one can consider closed intervals, since $C_{\mathrm{Stage}\,2}(\cdot)$ is continuous at end points.) Therefore,

$$\begin{aligned}C_{\mathrm{Stage}\,2}(\bar{Q}^{\mathrm{f}}) &= \theta^1\theta^2\, C_{\mathrm{U,U}}(\bar{Q}^{\mathrm{f}}) + \theta^1(1-\theta^2)C_{\mathrm{U,D}}(\bar{Q}^{\mathrm{f}}) \\ &\quad + (1-\theta^1)\theta^2\, C_{\mathrm{D,U}}(\bar{Q}^{\mathrm{f}}) + (1-\theta^1)(1-\theta^2) \\ &\quad \times C_{\mathrm{D,D}}(\bar{Q}^{\mathrm{f}}) \\ &= \Gamma(\bar{Q}^{\mathrm{f}}),\end{aligned}$$

and

$$\begin{aligned}\bar{Q}_6^{\mathrm{f}} &= \underset{\bar{Q}^{\mathrm{f}} \in I_6}{\arg\min}\ u^{\mathrm{f}}\, \bar{Q}^{\mathrm{f}} + C_{\mathrm{Stage}\,2}(\bar{Q}^{\mathrm{f}}) \\ &= \underset{\bar{Q}^{\mathrm{f}} \in I_6}{\arg\min}\ u^{\mathrm{f}}\, \bar{Q}^{\mathrm{f}} + \Gamma(\bar{Q}^{\mathrm{f}}).\end{aligned}$$

Next, if $\bar{Q}^{\mathrm{f}} \in I_5$ then, using part (3) of Theorem 1, $C_{\mathrm{U,U}}(\bar{Q}^{\mathrm{f}}) = (c^{\mathrm{f}} - c)\,\bar{Q}^{\mathrm{f}} + K_2$ for some constant K_2. Moreover, from Theorem 2 parts (3) and (2) (respectively) we have: $C_{\mathrm{U,D}}(\bar{Q}^{\mathrm{f}}) = (c_1^{\mathrm{f}} - c^1)\bar{Q}^{\mathrm{f}} + K_3$, and $C_{\mathrm{D,U}}(\bar{Q}^{\mathrm{f}}) = \Gamma(\bar{Q}^{\mathrm{f}})$. Also, from Theorem 3 part (2) $C_{\mathrm{D,D}} = \Gamma(\bar{Q}^{\mathrm{f}})$. Thus,

$$C_{\mathrm{Stage}\,2}(\bar{Q}^{\mathrm{f}}) = \left(-\theta^1\left(c^1 - c_1^{\mathrm{f}}\right)\right)\bar{Q}^{\mathrm{f}} + \bar{\theta}^1\Gamma(\bar{Q}^{\mathrm{f}}) + K,$$

for some constant K. Hence,

$$\bar{Q}_5^{\mathrm{f}} = \underset{\bar{Q}^{\mathrm{f}} \in I_5}{\arg\min}\left(u^{\mathrm{f}} - \theta^1\left(c^1 - c_1^{\mathrm{f}}\right)\right)\bar{Q}^{\mathrm{f}} + \bar{\theta}^1\Gamma(\bar{Q}^{\mathrm{f}}).$$

Next, if $\bar{Q}^{\mathrm{f}} \in I_4$ then, using part (3) of Theorems 1 and 2, $C_{\mathrm{U,U}}(\bar{Q}^{\mathrm{f}}) = (c^{\mathrm{f}} - c)\,\bar{Q}^{\mathrm{f}} + K_2$, $C_{\mathrm{D,U}}(\bar{Q}^{\mathrm{f}}) = (c_2^{\mathrm{f}} - c^2)\bar{Q}^{\mathrm{f}} + K_4$ (for some constants K_2, K_3, K_4). Also, from Theorem 3 part (2) $C_{\mathrm{D,D}} = \Gamma(\bar{Q}^{\mathrm{f}})$. Hence,

$$\begin{aligned}\bar{Q}_4^{\mathrm{f}} = \underset{\bar{Q}^{\mathrm{f}} \in I_4}{\arg\min}&\left(u^{\mathrm{f}} - \theta^1\left(c^1 - c_1^{\mathrm{f}}\right) - \bar{\theta}^1\theta^2\left(c^2 - c_2^{\mathrm{f}}\right)\right) \\ &\times \bar{Q}^{\mathrm{f}} + \bar{\theta}^1\bar{\theta}^2\Gamma(\bar{Q}^{\mathrm{f}}).\end{aligned}$$

Similarly, on I_3, from part (4) of Theorems 1 and 2: $C_{\mathrm{U,U}}(\bar{Q}^{\mathrm{f}}) = c_2^{\mathrm{f}}\bar{Q}^{\mathrm{f}} + G_2(\bar{Q}^{\mathrm{f}}) + K_5$, $C_{\mathrm{U,D}}(\bar{Q}^{\mathrm{f}}) = c_2^{\mathrm{f}}\bar{Q}^{\mathrm{f}} +$

$G_2(\bar{Q}^{\mathrm{f}}) + K_6$. Moreover, from part (3) of Theorem 2, $C_{\mathrm{D,U}}(\bar{Q}^{\mathrm{f}}) = (c_2^{\mathrm{f}} - c^2)\bar{Q}^{\mathrm{f}} + K_4$. Also, from Theorem 3 part (2), $C_{\mathrm{D,D}} = \Gamma(\bar{Q}^{\mathrm{f}})$. Hence,

$$\bar{Q}_3^{\mathrm{f}} = \arg\min_{\bar{Q}^{\mathrm{f}} \in I_3} \big(u^{\mathrm{f}} - \bar{\theta}^1\theta^2(c^2 - c_2^{\mathrm{f}}) + \theta^1 c_2^{\mathrm{f}}\big)\bar{Q}^{\mathrm{f}} + \theta^1 G_2(\bar{Q}^{\mathrm{f}}) + \bar{\theta}^1\bar{\theta}^2\Gamma(\bar{Q}^{\mathrm{f}}).$$

Next, on I_2, from part (5) of Theorem 1, $C_{\mathrm{U,U}}(\bar{Q}^{\mathrm{f}}) = (c_2^{\mathrm{f}} - c^2)\bar{Q}^{\mathrm{f}} + K_7$. Also, from Theorem 2, $C_{\mathrm{U,D}}(\bar{Q}^{\mathrm{f}}) = c_2^{\mathrm{f}}\bar{Q}^{\mathrm{f}} + G_2(\bar{Q}^{\mathrm{f}}) + K_6$. Moreover, from part (3) of Theorem 2, $C_{\mathrm{D,U}}(\bar{Q}^{\mathrm{f}}) = (c_2^{\mathrm{f}} - c^2)\bar{Q}^{\mathrm{f}} + K_4$. Also, from Theorem 3 part (2), $C_{\mathrm{D,D}} = \Gamma(\bar{Q}^{\mathrm{f}})$. Thus,

$$\bar{Q}_2^{\mathrm{f}} = \arg\min_{\bar{Q}^{\mathrm{f}} \in I_2} \big(u^{\mathrm{f}} - \theta^2(c^2 - c_2^{\mathrm{f}}) + \theta^1\bar{\theta}^2 c_2^{\mathrm{f}}\big)\bar{Q}^{\mathrm{f}} + \theta^1\bar{\theta}^2 G_2(\bar{Q}^{\mathrm{f}}) + \bar{\theta}^1\bar{\theta}^2\Gamma(\bar{Q}^{\mathrm{f}}).$$

Finally, on I_1, from part (5) of Theorem 1, $C_{\mathrm{U,U}}(\bar{Q}^{\mathrm{f}}) = (c_2^{\mathrm{f}} - c^2)\bar{Q}^{\mathrm{f}} + K_7$. Also, from Theorem 2, part (4), $C_{\mathrm{U,D}}(\bar{Q}^{\mathrm{f}}) = c_2^{\mathrm{f}}\bar{Q}^{\mathrm{f}} + G_2(\bar{Q}^{\mathrm{f}}) + K_6$ and $C_{\mathrm{D,U}}(\bar{Q}^{\mathrm{f}}) = c_1^{\mathrm{f}}\bar{Q}^{\mathrm{f}} + G_1(\bar{Q}^{\mathrm{f}}) + K_8$. Moreover, from Theorem 3 part (2), $C_{\mathrm{D,D}} = \Gamma(\bar{Q}^{\mathrm{f}})$. Thus,

$$\bar{Q}_1^{\mathrm{f}} = \arg\min_{\bar{Q}^{\mathrm{f}} \in I_1} \big(u^{\mathrm{f}} - \theta^1\theta^2(c^2 - c_2^{\mathrm{f}}) + \theta^1\bar{\theta}^2 c_2^{\mathrm{f}} + \bar{\theta}^1\theta^2 c_1^{\mathrm{f}}\big)\bar{Q}^{\mathrm{f}} + \bar{\theta}^1\theta^2 G_1(\bar{Q}^{\mathrm{f}}) + \theta^1\bar{\theta}^2 G_2(\bar{Q}^{\mathrm{f}}) + \bar{\theta}^1\bar{\theta}^2\Gamma(\bar{Q}^{\mathrm{f}}),$$

which completes the proof. ■

Proof of Corollary 1. To prove part (i) notice that, from Proposition 3, $\bar{Q}^{\mathrm{f}*} \in \{\bar{Q}_k^{\mathrm{f}}, k = 1, \ldots, 6\}$. Moreover, $\bar{Q}_k^{\mathrm{f}} \in I_k$ (see Proposition 3 for definition of I_k for $k = 1, \ldots, 6$). Thus, since every member of I_6 is greater than or equal to any member of I_k for $k = 1, \ldots, 5$, we have:

$$\bar{Q}^{\mathrm{f}*} \le \bar{Q}_6^{\mathrm{f}} \le \max_{\bar{Q}^{\mathrm{f}} \in I_6} \bar{Q}^{\mathrm{f}} = \sum_{j=1,2} F_j^{-1}\left(\frac{p_j + r_j - c_j^{\mathrm{f}}}{p_j + r_j + h_j}\right).$$

To prove part (ii) notice that, when $\theta^j(c^j - c_j^{\mathrm{f}}) \ge u^{\mathrm{f}}$ (for $j = 1, 2$), the optimal capacity reservation level without recourse presented in Theorem 7 is greater than or equal to the upper bound of capacity reservation level with recourse obtained in part (i) (since CDFs $F_j(.)$ are non-decreasing). Hence, when $\theta^j(c^j - c_j^{\mathrm{f}}) \ge u^{\mathrm{f}}$ (for $j = 1, 2$), the optimal capacity reservation level with recourse is less than or equal to the optimal capacity reservation level without recourse. ■

Proof of Theorem 4. Notice that when parameters are product independent, each of the intervals I_2, I_3, and I_5 in Proposition 3 only include a single point. Moreover, in that case, $I_2 = I_3 \subset I_4$ and $I_5 \subset I_6$. Thus, by Proposition 3, $\bar{Q}^{\mathrm{f}*} \in \{\bar{Q}_1^{\mathrm{f}}, \bar{Q}_4^{\mathrm{f}}, \bar{Q}_6^{\mathrm{f}}\}$. Additionally, using symmetric parameters in Proposition 3, $\Gamma(\bar{Q}^{\mathrm{f}}) = 2[c^{\mathrm{f}}\bar{Q}^{\mathrm{f}}/2 + G(\bar{Q}^{\mathrm{f}}/2)] = c^{\mathrm{f}}\bar{Q}^{\mathrm{f}} + 2G(\bar{Q}^{\mathrm{f}}/2)$.

Next notice that since $G(\cdot)$ is a convex function ($G''(\cdot) = (p + r + h)f(\cdot) \ge 0$ using Leibniz rule), $\Gamma(\cdot)$ is a convex function. Thus, $\bar{Q}_1^{\mathrm{f}}$, $\bar{Q}_4^{\mathrm{f}}$, and $\bar{Q}_6^{\mathrm{f}}$ are minimizers of convex functions on convex (and compact) sets. Thus, we can characterize them using the first order condition. We have:

$$\bar{Q}_1^{\mathrm{f}} = \arg\min_{\bar{Q}^{\mathrm{f}} \in I_1} \big(u^{\mathrm{f}} - \theta^1\theta^2(c^2 - c_2^{\mathrm{f}}) + \theta^1\bar{\theta}^2 c_2^{\mathrm{f}} + \bar{\theta}^1\theta^2 c_1^{\mathrm{f}}\big)\bar{Q}^{\mathrm{f}} + \bar{\theta}^1\theta^2 G_1(\bar{Q}^{\mathrm{f}}) + \theta^1\bar{\theta}^2 G_2(\bar{Q}^{\mathrm{f}}) + \bar{\theta}^1\bar{\theta}^2\Gamma(\bar{Q}^{\mathrm{f}}),$$

Therefore, to characterize $\bar{Q}_1^{\mathrm{f}}$, using the Leibniz rule and setting the derivative of the objective function equal to zero results in the candidate:

$$\hat{Q}_1^{\mathrm{f}} = \frac{d\big((1 - \theta^1\theta^2)(p + r) + \theta^1\theta^2 c - u^{\mathrm{f}} - c^{\mathrm{f}}\big)}{(\bar{\theta}^1\theta^2 + \theta^1\bar{\theta}^2 + (\bar{\theta}^1\bar{\theta}^2)/2)(p + r + h)}.$$

However, $\hat{Q}_1^{\mathrm{f}} \in I_1$ only if C1 does not hold but C2 holds, $\hat{Q}_1^{\mathrm{f}}$ is less than any point in I_1 if C2 does not hold, and $\hat{Q}_1^{\mathrm{f}}$ is greater than any point in I_1 otherwise. Thus,

$$\bar{Q}_1^{\mathrm{f}} = \begin{cases} 0 & \text{if C2 does not hold,} \\ \hat{Q}_1^{\mathrm{f}} & \text{if C2 holds but C1 does not hold,} \\ d(p + r - c)/(p + r + h) & \\ & \text{otherwise.} \end{cases} \tag{A28}$$

To characterize $\bar{Q}_4^{\mathrm{f}}$, notice that:

$$\bar{Q}_4^{\mathrm{f}} = \arg\min_{\bar{Q}^{\mathrm{f}} \in I_4} \big(u^{\mathrm{f}} - \theta^1(c^1 - c_1^{\mathrm{f}}) - \bar{\theta}^1\theta^2(c^2 - c_2^{\mathrm{f}})\big)\bar{Q}^{\mathrm{f}} + \bar{\theta}^1\bar{\theta}^2\Gamma(\bar{Q}^{\mathrm{f}}).$$

Thus, using the Leibniz rule and setting the derivative of the objective function equal to zero results in the candidate:

$$\hat{Q}_4^{\mathrm{f}} = \frac{2d\,(p + r - c - (u^{\mathrm{f}} - (c - c^{\mathrm{f}}))/(\bar{\theta}^1\bar{\theta}^2))}{p + r + h}.$$

However, $\hat{Q}_4^{\mathrm{f}}$ is in I_4 only if C1 holds and $c - c^{\mathrm{f}} < u^{\mathrm{f}}$, $\hat{Q}_4^{\mathrm{f}}$ is less than any point in I_4 if C1 does not hold and $c - c^{\mathrm{f}} < u^{\mathrm{f}}$, and $\hat{Q}_4^{\mathrm{f}}$ is greater than any point in I_4 if $c - c^{\mathrm{f}} \ge u^{\mathrm{f}}$. Thus,

$$\bar{Q}_4^{\mathrm{f}} = \begin{cases} d(p + r - c)/(p + r + h) & \\ & \text{if C1 does not hold and } c - c^{\mathrm{f}} < u^{\mathrm{f}}, \\ \hat{Q}_4^{\mathrm{f}} & \text{if C1 holds and } c - c^{\mathrm{f}} < u^{\mathrm{f}}, \\ 2d(p + r - c)/(p + r + h) & \\ & \text{if } c - c^{\mathrm{f}} \ge u^{\mathrm{f}}. \end{cases} \tag{A29}$$

Similarly, to characterize $\bar{Q}_6^{\mathrm{f}}$, using the Leibniz rule and setting the derivative of the objective function equal to zero results in the candidate:

$$\hat{Q}_6^{\mathrm{f}} = \frac{2d\,(p + r - c - (u^{\mathrm{f}} - (c - c^{\mathrm{f}}))/(\bar{\theta}^1\bar{\theta}^2))}{p + r + h}. \tag{A30}$$

However, $\hat{Q}_6^{\mathrm{f}} \in I_6$ only if $c - c^{\mathrm{f}} \ge u^{\mathrm{f}}$, $\hat{Q}_6^{\mathrm{f}}$ is less than any point in I_4 if $c - c^{\mathrm{f}} < u^{\mathrm{f}}$ and $\hat{Q}_6^{\mathrm{f}}$ cannot be greater than all points in I_6 (since $u^{\mathrm{f}} \geq 0$). Thus,

$$\bar{Q}_6^{\mathrm{f}} = \begin{cases} 2\,d\,(p+r-u^{\mathrm{f}}-c^{\mathrm{f}})/(p+r+h) & \text{if } c-c^{\mathrm{f}} \geq u^{\mathrm{f}}, \\ 2\,d\,(p+r-c)/(p+r+h) & \text{if } c-c^{\mathrm{f}} < u^{\mathrm{f}}. \end{cases} \tag{A31}$$

Now, since by Proposition 3 $\bar{Q}^{\mathrm{f}*} \in \{\bar{Q}_1^{\mathrm{f}}, \bar{Q}_4^{\mathrm{f}}, \bar{Q}_6^{\mathrm{f}}\}$, it remains to compare the cost of $\bar{Q}_1^{\mathrm{f}}$, $\bar{Q}_4^{\mathrm{f}}$, and $\bar{Q}_6^{\mathrm{f}}$ under different conditions. First, if $c-c^{\mathrm{f}} \geq u^{\mathrm{f}}$, then C1 and C2 trivially hold. Therefore, $\bar{Q}_1^{\mathrm{f}} = d\,(p+r-c)/(p+r+h)$ (from Equation (A28)), $\bar{Q}_4^{\mathrm{f}} = 2d\,(p+r-c)/(p+r+h)$ (from Equation (A29)), and $\bar{Q}_6^{\mathrm{f}} = 2d\,(p+r-u^{\mathrm{f}}-c^{\mathrm{f}})/(p+r+h)$ (from Equation (A31)). Thus, we notice that $I_1 = [0, \bar{Q}_1^{\mathrm{f}}]$, $I_4 = [\bar{Q}_1^{\mathrm{f}}, \bar{Q}_4^{\mathrm{f}}]$, and $I_6 = [\bar{Q}_4^{\mathrm{f}}, 2\,d\,(p+r-c)/(p+r+h)]$. However, since $\bar{Q}_4^{\mathrm{f}}$ is the minimizer over $I_4 = [\bar{Q}_1^{\mathrm{f}}, \bar{Q}_4^{\mathrm{f}}]$, $\bar{Q}_4^{\mathrm{f}}$ has a lower cost than $\bar{Q}_1^{\mathrm{f}}$. Moreover, since $\bar{Q}_6^{\mathrm{f}}$ is the optimizer over $I_6 = [\bar{Q}_4^{\mathrm{f}}, 2\,d\,(p+r-c)/(p+r+h)]$, $\bar{Q}_6^{\mathrm{f}}$ has a lower cost than $\bar{Q}_4^{\mathrm{f}}$. Hence, $\bar{Q}_6^{\mathrm{f}} = 2d\,(p+r-u^{\mathrm{f}}-c^{\mathrm{f}})/(p+r+h)$ is the optimal solution.

Second, consider the case where $c-c^{\mathrm{f}} < u^{\mathrm{f}}$. If as in (i) both C1 and C2 do not hold then $\bar{Q}_1^{\mathrm{f}} = 0$, $\bar{Q}_4^{\mathrm{f}} = d\,(p+r-c)/(p+r+h)$ and $\bar{Q}_6^{\mathrm{f}} = 2d\,(p+r-c)/(p+r+h)$. Next notice that $\bar{Q}_4^{\mathrm{f}}$ is a minimizer over $I_4 = [\bar{Q}_4^{\mathrm{f}}, \bar{Q}_6^{\mathrm{f}}]$ and hence has a lower cost than $\bar{Q}_6^{\mathrm{f}}$. Also, $\bar{Q}_1^{\mathrm{f}} = 0$ is a minimizer over $I_1 = [0, \bar{Q}_4^{\mathrm{f}}]$ and hence has a lower cost than $\bar{Q}_4$. Thus, in this case, $\bar{Q}^{\mathrm{f}*} = \bar{Q}_1^{\mathrm{f}} = 0$. However, if as in (ii) C1 does not hold but C2 holds then $\bar{Q}_1^{\mathrm{f}} = \hat{q}_1^{\mathrm{f}}$, $\bar{Q}_4^{\mathrm{f}} = d\,(p+r-c)/(p+r+h)$, and $\bar{Q}_6^{\mathrm{f}} = 2d\,(p+r-c)/(p+r+h)$. Next notice that $\bar{Q}_4^{\mathrm{f}}$ is a minimizer over $I_4 = [\bar{Q}_4^{\mathrm{f}}, \bar{Q}_6^{\mathrm{f}}]$ and hence has a lower cost than $\bar{Q}_6^{\mathrm{f}}$. Also, $\bar{Q}_1^{\mathrm{f}} = 0$ is a minimizer over $I_1 = [0, \bar{Q}_4^{\mathrm{f}}]$ and hence has a lower cost than $\bar{Q}_4$. Thus, in this case, $\bar{Q}^{\mathrm{f}*} = \hat{q}_1^{\mathrm{f}}$.

Next if as in (iii) C1 holds, first consider the case where C2 holds as well. In this case, $\bar{Q}_1^{\mathrm{f}} = d\,(p+r-c)/(p+r+h)$, $\bar{Q}_4^{\mathrm{f}} = \hat{Q}_4^{\mathrm{f}}$, and $\bar{Q}_6^{\mathrm{f}} = 2d\,(p+r-c)/(p+r+h)$. Next since $\bar{Q}_4^{\mathrm{f}} = \hat{Q}_4^{\mathrm{f}}$ is a minimizer over $I_4 = [\bar{Q}_1^{\mathrm{f}}, \bar{Q}_6^{\mathrm{f}}]$, $\bar{Q}_4^{\mathrm{f}} = \hat{Q}_4^{\mathrm{f}}$ has lower cost than $\bar{Q}_1^{\mathrm{f}}$ and $\bar{Q}_6^{\mathrm{f}}$. Thus, in this case $\bar{Q}^{\mathrm{f}*} = \hat{Q}_4^{\mathrm{f}}$. To complete part (iii) now suppose that C1 holds but C2 does not. Then $\bar{Q}_1^{\mathrm{f}} = 0$, $\bar{Q}_4^{\mathrm{f}} = \hat{Q}_4^{\mathrm{f}}$, and $\bar{Q}_6^{\mathrm{f}} = 2d\,(p+r-c)/(p+r+h)$. Next, similar to the previous case, since $\bar{Q}_6^{\mathrm{f}} \in I_4$ and $\bar{Q}_4^{\mathrm{f}}$ is a minimizer over I_4, $\bar{Q}_4^{\mathrm{f}} = \hat{Q}_4^{\mathrm{f}}$ has a lower cost than $\bar{Q}_6^{\mathrm{f}}$. Therefore, $\bar{Q}^{\mathrm{f}*} \in \{0, \hat{Q}_4^{\mathrm{f}}\}$. To determine $Q^{\mathrm{f}*}$ in this case, it is sufficient to compare the cost of options $\bar{Q}^{\mathrm{f}} = 0$ and $\bar{Q}^{\mathrm{f}} = \hat{Q}_4^{\mathrm{f}}$. To compute $C_{Stage2}(\bar{Q}_1^{\mathrm{f}} = 0)$, using part (5) of Theorem 1, part (4) of Theorem 2, and part (2) of Theorem 3 we have:

$$C_{\mathrm{U,U}}(0) = \frac{2\,cd(p+r-c)}{p+r+h} + 2G\left(\frac{d(p+r-c)}{p+r+h}\right),$$
$$C_{\mathrm{U,D}}(0) = \frac{cd(p+r-c)}{p+r+h} + G\left(\frac{d(p+r-c)}{p+r+h}\right) + G(0),$$
$$C_{\mathrm{D,U}}(0) = \frac{cd(p+r-c)}{p+r+h} + G\left(\frac{d(p+r-c)}{p+r+h}\right) + G(0),$$
$$C_{\mathrm{D,D}}(0) = 2\,G(0).$$

Hence,

$$\begin{aligned} C_{\mathrm{Stage}\,2}(0) &= \theta^1\,\theta^2\,C_{\mathrm{U,U}}(0) + \theta^1\,(1-\theta^2)C_{\mathrm{U,D}}(0) + (1-\theta^1) \\ &\quad \times \theta^2\,C_{\mathrm{D,U}}(0) + (1-\theta^1)(1-\theta^2)\,C_{\mathrm{D,D}}(0) \\ &= (\theta^1\bar{\theta}^2 + \bar{\theta}^1\theta^2 + 2\theta^1\theta^2)\left[\frac{cd(p+r-c)}{p+r+h}\right. \\ &\quad \left. + G\left(\frac{d(p+r-c)}{p+r+h}\right)\right] \\ &\quad + (\theta^1\bar{\theta}^2 + \bar{\theta}^1\theta^2 + 2\bar{\theta}^1\bar{\theta}^2)G(0). \end{aligned}$$

Moreover, since $\bar{Q}^{\mathrm{f}} = 0$, $u^{\mathrm{f}}\,\bar{Q}^{\mathrm{f}} + C_{\mathrm{Stage}\,2}(0) = C_{\mathrm{Stage}\,2}(0)$. Next, we compute the cost of option $\bar{Q}^{\mathrm{f}} = \hat{Q}_4^{\mathrm{f}}$ and compare it with the cost of $\bar{Q}^{\mathrm{f}} = 0$ computed above. Since $\hat{Q}_4^{\mathrm{f}} \in I_4$, we shall use part (3) of Theorems 1, 2, and part (1) of Theorem 3. Doing so we have:

$$\begin{aligned} C_{\mathrm{U,U}}(\hat{Q}_4^{\mathrm{f}}) &= (c^{\mathrm{f}}-c)\hat{Q}_4^{\mathrm{f}} + 2\left[\frac{cd(p+r-c)}{p+r+h} + G\left(\frac{d(p+r-c)}{p+r+h}\right)\right], \\ C_{\mathrm{U,D}}(\hat{Q}_4^{\mathrm{f}}) &= (c^{\mathrm{f}}-c)\hat{Q}_4^{\mathrm{f}} + 2\left[\frac{cd(p+r-c)}{p+r+h} + G\left(\frac{d(p+r-c)}{p+r+h}\right)\right], \\ C_{\mathrm{D,U}}(\hat{Q}_4^{\mathrm{f}}) &= (c^{\mathrm{f}}-c)\hat{Q}_4^{\mathrm{f}} + 2\left[\frac{cd(p+r-c)}{p+r+h} + G\left(\frac{d(p+r-c)}{p+r+h}\right)\right], \\ C_{\mathrm{D,D}}(\hat{Q}_4^{\mathrm{f}}) &= c^{\mathrm{f}}\,\hat{Q}_4^{\mathrm{f}} + 2G(\hat{Q}_4^{\mathrm{f}}/2). \end{aligned}$$

Hence,

$$\begin{aligned} C_{\mathrm{Stage}\,2}(\hat{Q}_4^{\mathrm{f}}) &= \theta^1\,\theta^2\,C_{U,U}(\hat{Q}_4^{\mathrm{f}}) + \theta^1\,(1-\theta^2)C_{\mathrm{U,D}}(\hat{Q}_4^{\mathrm{f}}) \\ &\quad + (1-\theta^1)\,\theta^2\,C_{\mathrm{D,U}}(\hat{Q}_4^{\mathrm{f}}) + (1-\theta^1)(1-\theta^2) \\ &\quad \times C_{\mathrm{D,D}}(\hat{Q}_4^{\mathrm{f}}) \\ &= [c^{\mathrm{f}} - (1-\bar{\theta}^1\bar{\theta}^2)c]\hat{Q}_4^{\mathrm{f}} + 2(1-\bar{\theta}^1\bar{\theta}^2) \\ &\quad \times \left[\frac{cd(p+r-c)}{p+r+h} + G\left(\frac{d(p+r-c)}{p+r+h}\right)\right] \\ &\quad + 2\bar{\theta}^1\bar{\theta}^2 G(\hat{Q}_4^{\mathrm{f}}/2), \end{aligned}$$

and the total optimal cost with $\bar{Q}^{\mathrm{f}} = \hat{Q}_4^{\mathrm{f}}$ is $u^{\mathrm{f}}\,\hat{Q}_4^{\mathrm{f}} + C_{\mathrm{Stage}\,2}(\hat{Q}_4^{\mathrm{f}})$. Thus, denoting the optimal cost with $\bar{Q}^{\mathrm{f}} = \hat{Q}_4^{\mathrm{f}}$ minus the optimal cost with $\bar{Q}^{\mathrm{f}} = 0$ by Δ (after simplification) we have:

$$\Delta = \big((1-\theta^1\theta^2)+\bar{\theta}^1\bar{\theta}^2\big)\left[\frac{cd(p+r-c)}{p+r+h} + G\left(\frac{d(p+r-c)}{p+r+h}\right) - G(0)\right] \tag{A32}$$

$$+\big(u^{\mathrm{f}}+c^{\mathrm{f}}-(1-\bar{\theta}^1\bar{\theta}^2)c\big)\hat{Q}_4^{\mathrm{f}} + 2\bar{\theta}^1\bar{\theta}^2\big[G\big(\hat{Q}_4^{\mathrm{f}}/2\big)-G(0)\big]. \tag{A33}$$

Next notice that both Equation (A32) and Equation (A33) are non-positive. To see that Equation (A32) is non-positive, define $g(x) = cx - G(x) - G(0)$ and notice that Equation (A32) is equal to $((1-\theta^1\theta^2)+\bar{\theta}^1\bar{\theta}^2)g(q^*)$ with

$$q^* = F^{-1}\left(\frac{p+r-c}{p+r+h}\right) = \frac{d(p+r-c)}{p+r+h}.$$

Hence, Equation (A32) is non-positive, since $g(0) = 0$ and q^* is the minimizer of $g(\cdot)$ ($G(\cdot)$ is convex and q^* is the solution to the first-order condition). Next, to see that Equation (A33) is non-positive, define:

$$\hat{g}(x) = (u^{\mathrm{f}}+c^{\mathrm{f}}-(1-\bar{\theta}^1\bar{\theta}^2)c)x + 2\bar{\theta}^1\bar{\theta}^2[G(x/2)-G(0)],$$

and notice that Equation (A33) is equal to $\hat{g}(\hat{Q}_4^{\mathrm{f}})$. Thus, Equation (A33) is non-positive, since $\hat{g}(0) = 0$ and $\hat{g}(\cdot)$ is a convex function minimized at $\hat{Q}_4^{\mathrm{f}}$ (see the definition of $\hat{Q}_4^{\mathrm{f}}$ or check the first and second-order conditions of $\hat{g}(\cdot)$). Hence, $\Delta \le 0$. Thus, $\bar{Q}^{\mathrm{f}*} = \hat{Q}_4^{\mathrm{f}}$, and the proof is complete. ■

Proof of Theorem 5. Notice that program (12) is convex with linear constraints, and therefore KKT conditions are necessary and sufficient to characterize the optimal solution. Assume that λ_j^{f} $(j = 1, 2)$ and μ represent the Lagrangian multipliers corresponding to constraints $q_j^{\mathrm{f}} \ge 0$ and $q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \le \bar{Q}^{\mathrm{f}}$, respectively. Using the Leibniz rule, KKT conditions are

$$q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \le \bar{Q}^{\mathrm{f}},$$

$$c_j^{\mathrm{f}} + (h_j+p_j+r_j)F_j\big(q^j+q_j^{\mathrm{f}}\big) - p_j - r_j = \lambda_j^{\mathrm{f}} - \mu \quad (j=1,2), \tag{A34}$$

$$\mu\big(q_1^{\mathrm{f}}+q_2^{\mathrm{f}}-\bar{Q}^{\mathrm{f}}\big) = 0,$$

$$\lambda_j^{\mathrm{f}} q_j^{f} = 0, \quad (j=1,2),$$

$$q_j^{\mathrm{f}}, \mu, \lambda_j^{f} \ge 0. \quad (j=1,2).$$

Condition (A34) results in

$$q_j^{\mathrm{f}} = F_j^{-1}\left(\frac{p_j+r_j-c_j^{\mathrm{f}}+\lambda_j^{\mathrm{f}}-\mu}{p_j+r_j+h_j}\right) - q^j. \quad (j=1,2).$$

Hence, the KKT conditions can be written as follows:

$$q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \le \bar{Q}^{\mathrm{f}}, \tag{A35}$$

$$q_j^{\mathrm{f}} = F_j^{-1}\left(\frac{p_j+r_j-c_j^{\mathrm{f}}+\lambda_j^{\mathrm{f}}-\mu}{p_j+r_j+h_j}\right) - q^j \quad (j=1,2), \tag{A36}$$

$$\mu\big(q_1^{\mathrm{f}}+q_2^{\mathrm{f}}-\bar{Q}^{\mathrm{f}}\big) = 0, \tag{A37}$$

$$\lambda_j^{\mathrm{f}} q_j^{f} = 0 \quad (j=1,2), \tag{A38}$$

$$q_j^{\mathrm{f}}, \mu, \lambda_j^{f} \ge 0. \quad (j=1,2). \tag{A39}$$

Now, it is trivial to show that the optimal solutions α_j and β_j satisfy conditions (A35) to (A39) in the appropriate range of $\bar{Q}^{\mathrm{f}}$ defined in the theorem. ■

Proof of Theorem 6. The proof follows from Theorem 5 after setting $q^n = 0$. ■

Proof of Theorem 7. Let μ, λ^j, and λ_j^{f} for $(j = 1, 2)$ denote the Lagrangian multipliers, respectively, for constraints (A65) to (A66). KKT conditions (with subscript P denoting the perceived costs) are then:

$$q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \le \bar{Q}^{\mathrm{f}}, \tag{A40}$$

$$\partial C_P/\partial q^j - \lambda^j = 0 \quad (j=1,2), \tag{A41}$$

$$\partial C_P/\partial q_j^{\mathrm{f}} - \lambda_j^{\mathrm{f}} + \mu = 0 \quad (j=1,2), \tag{A42}$$

$$\partial C_P/\partial \bar{Q}^{\mathrm{f}} - \mu = 0 \quad (j=1,2), \tag{A43}$$

$$\mu\big(q_1^{\mathrm{f}}+q_2^{\mathrm{f}}-\bar{Q}^{\mathrm{f}}\big) = 0, \tag{A44}$$

$$\lambda^j q^j = 0 \quad (j=1,2), \tag{A45}$$

$$\lambda_j^{\mathrm{f}} q_j^{f} = 0 \quad (j=1,2), \tag{A46}$$

$$q^j, q_j^{\mathrm{f}}, \mu, \lambda^j, \lambda_j^{f} \ge 0 \quad (j=1,2). \tag{A47}$$

Moreover, using the Leibniz rule we have:

$$\partial C_{\mathrm{P}}/\partial q^j = \theta^j[c^j + (h_j+p_j+r_j)F_j\big(q^j+q_j^{\mathrm{f}}\big) - p_j - r_j] \quad (j=1,2),$$

$$\partial C_{\mathrm{P}}/\partial q_j^{\mathrm{f}} = c_j^{\mathrm{f}} + \theta^j[(h_j+p_j+r_j)F_j\big(q^j+q_j^{\mathrm{f}}\big) - p_j - r_j], + (1-\theta^j)[(h_j+p_j+r_j)F_j\big(q_j^{\mathrm{f}}\big) - p_j - r_j] \quad (j=1,2),$$

$$\partial C_{\mathrm{P}}/\partial \bar{Q}^{\mathrm{f}} = u_j^{\mathrm{f}}. \quad (j=1,2).$$

Hence, conditions (A40) to (A47) are

$$
\begin{aligned}
&q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \le \bar{Q}^{\mathrm{f}},\\
&\theta^j[c^j + (h_j + p_j + r_j)F_j(q^j + q_j^{\mathrm{f}}) - p_j - r_j] = \lambda^j \quad (j = 1, 2),\\
&c_j^{\mathrm{f}} + \theta^j[(h_j + p_j + r_j)F_j(q^j + q_j^{\mathrm{f}}) - p_j - r_j] + (1 - \theta^j)\\
&\quad [(h_j + p_j + r_j)F_j(q_j^{\mathrm{f}}) - p_j - r_j] = \lambda_j^{\mathrm{f}} - \mu \quad (j = 1, 2),\\
&\mu = u_j^{\mathrm{f}} \quad (j = 1, 2),\\
&\mu(q_1^{\mathrm{f}} + q_2^{\mathrm{f}} - \bar{Q}^{\mathrm{f}}) = 0\\
&\lambda^j q^j = 0 \quad (j = 1, 2),\\
&\lambda_j^{\mathrm{f}} q_j^{\mathrm{f}} = 0 \quad (j = 1, 2),\\
&q^j, q_j^{\mathrm{f}}, \mu, \lambda^j, \lambda_j^{\mathrm{f}} \ge 0 \quad (j = 1, 2).
\end{aligned}
$$

or equivalently:

$$F_j(q^{j*} + q_j^{\mathrm{f}*}) = \frac{\lambda^j/\theta^j + p_j + r_j - c^j}{p_j + r_j + h_j} \quad (j = 1, 2), \quad \text{(A48)}$$

$$F_j(Q_j^{\mathrm{f}*}) = \frac{(\lambda_j^{\mathrm{f}} - \lambda^j + \theta^j c^j - u_j^{\mathrm{f}})/(1 - \theta^j) + p_j + r_j}{p_j + r_j + h_j} \quad (j = 1, 2), \quad \text{(A49)}$$

$$
\begin{aligned}
&\bar{Q}^{\mathrm{f}*} = q_1^{\mathrm{f}*} + q_2^{\mathrm{f}*}\\
&\lambda^j q^{j*} = 0 \quad (j = 1, 2), \quad \text{(A50)}\\
&\lambda_j^{\mathrm{f}} q_j^{\mathrm{f}*} = 0 \quad (j = 1, 2), \quad \text{(A51)}\\
&q^j, q_j^{\mathrm{f}}, \lambda^j, \lambda_j^{\mathrm{f}} \ge 0, \quad (j = 1, 2). \quad \text{(A52)}
\end{aligned}
$$

Since by assumption $u_j^{\mathrm{f}} > c^j$ and $h_j \ge 0$, we have

$$\frac{\big((\theta^j c^j - u_j^{\mathrm{f}})/(1 - \theta^j)\big) + p_j + r_j}{p_j + r_j + h_j} \le 1.$$

Hence, if

$$\frac{\big((\theta^j c^j - u_j^{\mathrm{f}})/(1 - \theta^j)\big) + p_j + r_j}{p_j + r_j + h_j} \ge 0,$$

setting $\lambda^j = \lambda_j^{\mathrm{f}} = 0$ (for every $j \in \{1, 2\}$ that satisfies this inequality) to guarantee conditions (A50) to (A52) results in the optimal solution:

$$
\begin{aligned}
q_j^{\mathrm{f}*} &= F_j^{-1}\left(\frac{p_j+r_j-\big((u_j^{\mathrm{f}}-\theta^j c^j)/(1-\theta^j)\big)}{p_j+r_j+h_j}\right) \quad (j = 1, 2)\\
q^{j*} &= F_j^{-1}\left(\frac{p_j + r_j - c^j}{p_j + r_j + h_j}\right)\\
&\quad - F_j^{-1}\left(\frac{p_j+r_j-\big((u_j^{\mathrm{f}}-\theta^j c^j)/(1-\theta^j)\big)}{p_j+r_j+h_j}\right) \quad (j = 1, 2)\\
\bar{Q}^{\mathrm{f}*} &= \sum_{j=1}^{2} Q_j^{\mathrm{f}*} = \sum_{j=1}^{2} F_j^{-1}\left(\frac{p_j+r_j-\big((u_j^{\mathrm{f}}-\theta^j c^j)/(1-\theta^j)\big)}{p_j+r_j+h_j}\right).
\end{aligned}
$$

However, if

$$\frac{\big((\theta^j c^j - u_j^{\mathrm{f}})/(1 - \theta^j)\big) + p_j + r_j}{p_j + r_j + h_j} < 0,$$

Equation (A49) cannot be satisfied with $\lambda^j = \lambda_j^{\mathrm{f}} = 0$. In this case, however, setting $Q_j^{\mathrm{f}*} = \lambda^j = 0$ satisfies all KKT conditions and hence is optimal. Therefore, because $F_j(0) = 0$, a general optimal solution for $0 \le \theta^j < 1$ is

$$
\begin{aligned}
Q_j^{\mathrm{f}*} &= F_j^{-1}\left(\left[\frac{p_j + r_j - \big((u_j^{\mathrm{f}} - \theta^j c^j)/(1 - \theta^j)\big)}{p_j + r_j + h_j}\right]^+\right) \quad (j = 1, 2),\\
q^{j*} &= F_j^{-1}\left(\frac{p_j + r_j - c^j}{p_j + r_j + h_j}\right)\\
&\quad - F_j^{-1}\left(\left[\frac{p_j+r_j-\big((u_j^{\mathrm{f}}-\theta^j c^j)/(1-\theta^j)\big)}{p_j+r_j+h_j}\right]^+\right) \quad (j = 1, 2),\\
\bar{Q}^{\mathrm{f}*} &= \sum_{j=1}^{2} Q_j^{\mathrm{f}*} = \sum_{j=1}^{2} F_j^{-1}\left(\left[\frac{p_j+r_j-\big((u_j^{\mathrm{f}}-\theta^j c^j)/(1-\theta^j)\big)}{p_j+r_j+h_j}\right]^+\right),
\end{aligned}
$$

and the proof is complete. ■

Proof of Lemma 1. To obtain optimal decisions in the absence of the flexible supplier, set $q_1^{\mathrm{f}} = q_2^{\mathrm{f}} = \bar{Q}^{\mathrm{f}} = 0$ in Equation (A63) and use the Leibniz rule to derive first-order condition (or simply use the results of a basic newsvendor model with lost sales). Doing that we get

$$q'^{j*} = F_j^{-1}\left(\frac{p_j + r_j - c^j}{p_j + r_j + h_j}\right).$$

Then, using Equation (A67) we have:

$$
\begin{aligned}
&C_{\mathrm{T}}(q'^{1*}, q'^{2*}, 0, 0, 0)\\
&= \sum_{j=1}^{2}\left[(1 - \pi_0^j)\left(c^j F_j^{-1}\left(\frac{p_j + r_j - c^j}{p_j + r_j + h_j}\right)\right.\right.\\
&\quad \left.\left. + G_j\left(F_j^{-1}\left(\frac{p_j + r_j - c^j}{p_j + r_j + h_j}\right)\right)\right) + \pi_0^j G_j(0)\right].
\end{aligned}
$$

Also, to obtain $C_{\mathrm{T}}(q^{1*}, q^{2*}, Q_1^{\mathrm{f}*}, Q_2^{\mathrm{f}*}, \bar{Q}^{\mathrm{f}*})$, substitute the perceived optimal values (derived by Theorem 7) in to Equation (A67). Hence, using Equation (21) and after simplification we have:

$$
\begin{aligned}
V^{\mathrm{f}} &= C_{\mathrm{T}}(q'^{1*}, q'^{2*}, 0, 0, 0) - C_{\mathrm{T}}(q^{1*}, q^{2*}, Q_1^{\mathrm{f}*}, Q_2^{\mathrm{f}*}, \bar{Q}^{\mathrm{f}*})\\
&= \sum_{j=1}^{2}\left[\pi_0^j\,(G_j(0) - G_j(Q^{\mathrm{f}*})) - (u_j^{\mathrm{f}} - (1 - \pi_0^j)c^j)\,Q_j^{\mathrm{f}*}\right].
\end{aligned}
$$

Now, replacing $G_j(0) = p_j E(D_j)$ in the above equation completes the proof. ■

Proof of Theorem 8. To prove part (i), note that the value of the flexible supplier as perceived by the firm (and not its true value) is

$$V^{\mathrm{f}} = \sum_{j=1}^{2}(1 - \theta^j)\big(p_j E(D_j) - G_j(Q_j^{\mathrm{f}*})\big) - (u_j^{\mathrm{f}} - \theta^j c^j)\,Q_j^{\mathrm{f}*},$$

where for simplicity we have removed subscript P to denote that V^{f} here is a perceived value. We first show that V^{f} as perceived by the firm is positive if and only if

$$\exists j \in \{1,2\}: \; q_j^{\mathrm{f}*} = F_j^{-1}\left(\left[\frac{p_j + r_j - \left(\left(u_j^{\mathrm{f}} - \theta^j c^j\right)/(1-\theta^j)\right)}{p_j + r_j + h_j}\right]^+\right) > 0.$$

To show this, for $x \in [0, \infty)$ let $V_j^{\mathrm{f}}(x) = (1 - \theta^j)(p_j E(D_j) - G_j(x)) - (u_j^{\mathrm{f}} - \theta^j c^j)\,x$ denote the perceived value of the flexible supplier with respect to product j if the firm orders x units of product j from the flexible supplier. Then we have:

$$\frac{\partial\left(V_j^{\mathrm{f}}\right)}{\partial x} = -(1-\theta^j)((h_j + p_j + r_j)F_j(x) - p_j - r_j) + \theta^j c^j - u_j^{\mathrm{f}}.$$

Hence,

$$\frac{\partial}{\partial x}\left(\frac{\partial\left(V_j^{\mathrm{f}}\right)}{\partial x}\right) = -(1-\theta^j)\,(h_j + p_j + r_j) f_j(x) \le 0$$

Therefore, $V_j^{\mathrm{f}}(\cdot)$ is concave and the first-order condition yields

$$\max\left\{V_j^{\mathrm{f}}(x) : x \in [0,\infty)\right\} = V_j^{\mathrm{f}}\left(Q_j^{\mathrm{f}*}\right).$$

Additionally, because $G_j(0) = p_j E(D_j)$, we get $V_j^{\mathrm{f}}(0) = 0$. Hence, we have $V_j^{\mathrm{f}}(Q_j^{\mathrm{f}*}) \ge 0$. Also, $\partial(V_j^{\mathrm{f}})/\partial x > 0$ for $x \in [0, Q_j^{\mathrm{f}*})$ shows that $V_j^{\mathrm{f}}(\cdot)$ is increasing in $[0, Q_j^{\mathrm{f}*})$. Hence, $V_j^{\mathrm{f}}(Q_j^{\mathrm{f}*}) > 0$ if, and only if, $Q_j^{\mathrm{f}*} > 0$. Now, note that the value of the flexible supplier as perceived by the firm is $V^{\mathrm{f}} = \sum_{j=1}^{2} V_j^{\mathrm{f}}(Q_j^{\mathrm{f}*})$. Therefore, $V^{\mathrm{f}} > 0$ if, and only if,

$$\exists j \in \{1,2\}: \; Q_j^{\mathrm{f}*} = F_j^{-1}\left(\left[\frac{p_j + r_j - \left(\left(u_j^{\mathrm{f}} - \theta^j c^j\right)/(1-\theta^j)\right)}{p_j + r_j + h_j}\right]^+\right) > 0.$$

Moreover, because $F_j(\cdot)$ is non-decreasing and $F_j(x) > 0$ for all $x > 0$, the above condition is equivalent to

$$\exists j \in \{1,2\}: \; p_j + r_j - \frac{u_j^{\mathrm{f}} - \theta^j c^j}{1-\theta^j} > 0,$$

which is equivalent to

$$\exists j \in \{1,2\}: (1-\theta^j)(p_j + r_j) > u_j^{\mathrm{f}} - \theta^j c^j,$$

or, similarly

$$\exists j \in \{1,2\}: p_j + r_j - u_j^{\mathrm{f}} > \theta^j (p_j + r_j - c^j).$$

This is equivalent to Equation (23) (as $r_j > c^j$ and hence $(p_j + r_j - c^j) > 0$).

We prove part (ii) by contradiction. First, note that by Lemma 1, the true value of the flexible supplier for the firm is

$$V^{\mathrm{f}} = \sum_{j=1}^{2}\left[\pi_0^j\left(p_j E(D_j) - G_j\left(Q_j^{\mathrm{f}*}\right)\right) - u_j^{\mathrm{f}} - \left(1-\pi_0^j\right)c^j\, Q_j^{\mathrm{f}*}\right].$$

Now suppose $V^{\mathrm{f}} > 0$. Because $Q_1^{\mathrm{f}*} = Q_2^{\mathrm{f}*} = 0$ results in $V^{\mathrm{f}} = 0$, we then have: $\exists j \in \{1,2\}: \; Q_j^{\mathrm{f}*} > 0$ or equivalently (by proof of part (i)):

$$\exists j \in \{1,2\}: \; \theta^j < \frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j - c^j}.$$

Since the proof for other cases can be obtained in the same way (merely a change of notation), without loss of generality suppose this is true for $j = 1$; i.e., we have:

$$\theta^1 < \frac{p_1 + r_1 - u_1^{\mathrm{f}}}{p_1 + r_1 - c^1}, \tag{A53}$$

and $Q_2^{\mathrm{f}*} = 0$. We then show that it yields:

$$\max\left\{\theta^1, 1 - \pi_0^1\right\} < \frac{p_1 + r_1 - u_1^{\mathrm{f}}}{p_1 + r_1 - c^1}, \tag{A54}$$

which is a contradiction with the condition given in Equation (24) that for both $j = 1, 2$:

$$\max\left\{\theta^j, 1 - \pi_0^j\right\} \ge \frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j - c^j}.$$

To show that we get Equation (A54), note that using Equation (A53), we just need to show that we get:

$$1 - \pi_0^1 < \frac{p_1 + r_1 - u_j^{\mathrm{f}}}{p_1 + r_1 - c^1} \tag{A55}$$

for the case where $\theta^1 < 1 - \pi_0^1$. To show that we get Equation (A55) in this case, let $V_j^{\mathrm{f}}(x) = \pi_0^j(p_j E(D_j) - G_j(x)) - (u_j^{\mathrm{f}} - (1-\pi_0^j)c^j)\,x$ for $x \in [0,\infty)$ denote the true value of the flexible supplier with respect to product j if the firm orders x units of product j from the flexible supplier. Now, consider the fact that:

$$\frac{\partial\left(V_j^{\mathrm{f}}\right)}{\partial x} = -\pi_0^j((h_j + p_j + r_j)F_j(x) - p_j - r_j) + \left(1-\pi_0^j\right)c^j - u_j^{\mathrm{f}},$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial\left(V_j^{\mathrm{f}}\right)}{\partial x}\right) = -\pi_0^j\,(h_j + p_j + r_j) f_j(x) \le 0.$$

Hence, $V_j^{\mathrm{f}}(\cdot)$ defined above is concave and the first-order condition yields:

$$\max\left\{V_j^{\mathrm{f}}(x) : x \in [0,\infty)\right\} = V_j^{\mathrm{f}}\left(F_j^{-1}\left(\frac{p_j + r_j - \left(\left(u_j^{\mathrm{f}} - \left(1-\pi_0^j\right)c^j\right)/\left(\pi_0^j\right)\right)}{p_j + r_j + h_j}\right)\right).$$

Now, since $\theta^1 < 1-\pi_0^1$, we have:

$$\frac{p_j + r_j - \left(\left(u_j^{\mathrm{f}} - \left(1-\pi_0^j\right)c^j\right)/\pi_0^j\right)}{p_j + r_j + h_j} < \frac{p_j + r_j - \left(\left(u_j^{\mathrm{f}} - \theta^j c^j\right)/(1-\theta^j)\right)}{p_j + r_j + h_j} \leq F_j\left(q_j^{\mathrm{f}*}\right).$$

Therefore, as $F_j(\cdot)$ is non-decreasing, we have

$$F_j^{-1}\left(\frac{p_j + r_j - \left(\left(u_j^{\mathrm{f}} - \left(1-\pi_0^j\right)c^j\right)/\pi_0^j\right)}{p_j + r_j + h_j}\right) \leq q_j^{\mathrm{f}*}.$$

Furthermore, because $\partial(V_j^{\mathrm{f}})/\partial x < 0$ in the interval:

$$\left(F_j^{-1}\left(\frac{p_j + r_j - \left(\left(u_j^{\mathrm{f}} - \left(1-\pi_0^j\right)c^j\right)/\pi_0^j\right)}{p_j + r_j + h_j}\right), +\infty\right),$$

$V_j^{\mathrm{f}}(x)$ is strictly decreasing in this interval. Hence, considering the initial assumptions that $V^{\mathrm{f}} > 0$ and $Q_2^{\mathrm{f}*} = 0$ we have:

$$0 < V^{\mathrm{f}} = V_1^{\mathrm{f}}\left(q_1^{\mathrm{f}*}\right) + V_2^{\mathrm{f}}\left(q_2^{\mathrm{f}*}\right) < V_1^{\mathrm{f}}\left(F_j^{-1}\left(\frac{p_j + r_j - \left(\left(u_j^{\mathrm{f}} - \left(1-\pi_0^j\right)c^j\right)/\pi_0^j\right)}{p_j + r_j + h_j}\right)\right) + 0.$$

Now because $V_1^{\mathrm{f}}(0) = 0$ and $F_j(\cdot)$ is non-decreasing and $F_j(x) > 0$ for all $x > 0$, the above condition yields:

$$\frac{p_j + r_j - \left(\left(u_j^{\mathrm{f}} - \left(1-\pi_0^j\right)c^j\right)/\pi_0^j\right)}{p_j + r_j + h_j} > 0,$$

which is equivalent to Equation (A55), which in turn implies Equation (A54). However, this is a contradiction and hence the proof is complete. ■

Proof of Proposition 4. It is sufficient to show that for both $j = 1, 2$: $\partial(V^{\mathrm{f}})/\partial\epsilon^j \leq 0$ if $\epsilon^j > 0$ and $\partial(V^{\mathrm{f}})/\partial\epsilon^j \geq 0$ otherwise. To show this note that by Lemma 1 we have:

$$V^{\mathrm{f}}(\epsilon^1, \epsilon^2) = \sum_{j=1}^{2}\left[\pi_0^j\left(p_j E(D_j) - G_j\left(q_j^{\mathrm{f}*}\right)\right) - \left(u_j^{\mathrm{f}} - \left(1-\pi_0^j\right)c^j\right)q_j^{\mathrm{f}*}\right]$$

where using Theorem 7:

$$q_j^{\mathrm{f}*} = F_j^{-1}\left(\left[\frac{p_j + r_j - \left(\left(u_j^{\mathrm{f}} - \left(1-\pi_0^j + \epsilon^j\right)c^j\right)/(\pi_0 - \epsilon^j)\right)}{p_j + r_j + h_j}\right]^{+}\right). \tag{A56}$$

Hence, using the technique for computing derivative of an inverse function, we have:

$$\frac{\partial\left(q_j^{\mathrm{f}*}\right)}{\partial\epsilon^j} = -\frac{\left(u_j^{\mathrm{f}} - c^j\right)}{(p_j + r_j + h_j)(\pi_0 - \epsilon^j)^2 f_j\left(q_j^{\mathrm{f}*}\right)} \leq 0 \tag{A57}$$

since $u_j^{\mathrm{f}} \geq c^j$ for both $j = 1, 2$. Moreover,

$$\frac{\partial(V^{\mathrm{f}})}{\partial\epsilon^j} = \frac{\partial(V^{\mathrm{f}})}{\partial q_j^{\mathrm{f}*}}\frac{\partial q_j^{\mathrm{f}*}}{\partial\epsilon^j}.$$

Therefore,

$$\frac{\partial(V^{\mathrm{f}})}{\partial\epsilon^j} = -\frac{\left(u_j^{\mathrm{f}} - c^j\right)}{(p_j + r_j + h_j)(\pi_0 - \epsilon^j)^2 f_j\left(q_j^{\mathrm{f}*}\right)} \times \left[-\pi_0^j\left((p_j + r_j + h_j)F_j\left(q_j^{\mathrm{f}*}\right) - p_j - r_j\right) - \left(u_j^{\mathrm{f}} - (1-\pi_0)c^j\right)\right]. \tag{A58}$$

Now, notice that because of Equation (A56), $-\pi_0^j((p_j + r_j + h_j)F_j(q_j^{\mathrm{f}*}) - p_j - r_j) - (u_j^{\mathrm{f}} - (1-\pi_0)c^j)$ is not negative if $\epsilon^j \geq 0$ and is not positive if $\epsilon^j \leq 0$. Hence, using inequality (A57) and Equation (A58), the proof is complete. ■

Proof of Lemma 2. If the firm knows that dedicated supplier j is up, it would not reserve the flexible supplier's capacity for product j and would just order from dedicated supplier j because this supplier is cheaper than the flexible one. The optimal ordering quantity of this product in this case can be determined using a simple single-source newsvendor problem. This quantity is the same as total optimal ordering quantity for product j in the general no-information case; i.e., $q_j^{\mathrm{f}*} + q^{j*}$ where $q^{\mathrm{f}*}$ and q^{j*} are presented in Theorem 7. Using these optimal quantities we would have:

$$\begin{aligned} V^{+^j} &= C_{\mathrm{T}}\left(q^{1*}, q^{2*}, q_1^{\mathrm{f}*}, q_2^{\mathrm{f}*}, \bar{q}^{\mathrm{f}*}\right) \\ &\quad - C_{\mathrm{T}}\left(q^{1\#}, q^{2\#}, q_1^{\mathrm{f}\#}, q_2^{\mathrm{f}\#}, \bar{Q}^{\mathrm{f}\#}|i^j > 0\right) \\ &= u_j^{\mathrm{f}} Q_j^{\mathrm{f}*} + \left[(1-\pi_0^j)(c^j q^{j*} + G_j\left(q^{j*} + q_j^{\mathrm{f}*}\right)) + \pi_0^j\, G_j\left(q_j^{\mathrm{f}*}\right)\right] \\ &\quad - G_j\left(q_j^{\mathrm{f}*} + q^{j*}\right) - c^j\left(Q_j^{\mathrm{f}*} + q^{j*}\right). \end{aligned}$$

Then, simplification results in Equation (27). To derive V^{0^j}, note that if the firm knows that dedicated supplier j is in the down state (i.e., is disrupted), it procures only from the flexible supplier (for product j) and will solve a single-source newsvendor problem with procurement cost of u_j^{f}. Hence, in that case, its optimal cost for this product is

$$G_j\left(F_j^{-1}\left(\frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j + h_j}\right)\right) + u_j^{\mathrm{f}} F_j^{-1}\left(\frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j + h_j}\right).$$

Then, using Equation (25) and simplification yields Equation (28). Moreover, by implementing these values for V^{+^j} and V^{0^j}, the value of information on the threat level of dedicated supplier j (VI^j) can be obtained using its definition (i.e., $VI^j = (1-\pi_0^j)V^{+^j} + \pi_0^j V^{0^j}$). Using this and simplification then results in Equation (29) and the proof is complete. ■

Proof of Proposition 5. From Lemmas 1 and 2 (and after simplification) we have:

$$VI - V^{\mathrm{f}} = \sum_{j=1}^{2}\Bigg[2(u_j^{\mathrm{f}} - c^j)q_j^{\mathrm{f}*} + \pi_0^j\Bigg[2\left(G_j(q_j^{\mathrm{f}*}) + c^j q_j^{\mathrm{f}*}\right) - \left(G_j\left(F_j^{-1}\left(\frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j + h_j}\right)\right) + u_j^{\mathrm{f}} F_j^{-1}\left(\frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j + h_j}\right) + p_j E(D_j)\right)\Bigg]\Bigg].$$

Next if

$$2\left(G_j(q_j^{\mathrm{f}*}) + c^j q_j^{\mathrm{f}*}\right) - \left(G_j\left(F_j^{-1}\left(\frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j + h_j}\right)\right) + u_j^{\mathrm{f}} F_j^{-1}\left(\frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j + h_j}\right) + p_j E(D_j)\right) \geq 0,$$

let $\hat{\pi}_0^j = 1$. Otherwise, let

$$\hat{\pi}_0^j = \min\left\{\frac{-2\left(u_j^{\mathrm{f}} - c^j\right) q_j^{\mathrm{f}*}}{2\left(G_j(q_j^{\mathrm{f}*}) + c^j q_j^{\mathrm{f}*}\right) - \left(G_j(F_j^{-1}((p_j + r_j - u_j^{\mathrm{f}})/(p_j + r_j + h_j))) + u_j^{\mathrm{f}} F_j^{-1}((p_j + r_j - u_j^{\mathrm{f}})/(p_j + r_j + h_j)) + p_j E(D_j)\right)}, 1\right\},$$

and notice that since $u_j^{\mathrm{f}} \geq c^j$, $VI - V^{\mathrm{f}} \geq 0$ whenever $\pi_0^j \leq \hat{\pi}_0^j$. ■

Proof of Proposition 6. To show the first part, it is sufficient to show that for both $j = 1, 2$ $\partial(VI)/\partial\epsilon^j \leq 0$ if $\epsilon^j < 0$ and $\partial(VI)/\partial\epsilon^j \geq 0$ if $\epsilon^j > 0$. To show this note that:

$$\frac{\partial(VI)}{\partial\epsilon^j} = \frac{\partial(VI)}{\partial q_j^{\mathrm{f}*}}\frac{\partial q_j^{\mathrm{f}*}}{\partial\epsilon^j}.$$

Moreover, recall that:

$$q_j^{\mathrm{f}*} = F_j^{-1}\left(\left[\frac{p_j + r_j - \left(\left(u_j^{\mathrm{f}} - \left(1 - \pi_0^j + \epsilon^j\right)c^j\right)/(\pi_0 - \epsilon^j)\right)}{p_j + r_j + h_j}\right]^+\right). \tag{A59}$$

Hence, using the technique to get the derivative of an inverse function we have:

$$\frac{\partial\left(q_j^{\mathrm{f}*}\right)}{\partial\epsilon^j} = -\frac{\left(u_j^{\mathrm{f}} - c^j\right)}{(p_j + r_j + h_j)(\pi_0 - \epsilon^j)^2 f_j\left(q_j^{\mathrm{f}*}\right)} \leq 0, \tag{A60}$$

since $u_j^{\mathrm{f}} \geq c^j$ for both $j = 1, 2$. Therefore, to show the result, it is sufficient to show that $\partial(VI)/\partial Q_j^{\mathrm{f}*} \leq 0$ if $\epsilon^j > 0$ and $(\partial(VI))/(\partial Q_j^{\mathrm{f}*}) \geq 0$ if $\epsilon^j < 0$. But from Lemma 3 we have:

$$VI^j = (u_j^{\mathrm{f}} - \left(1 - \pi_0^j\right) c^j)q_j^{\mathrm{f}*} + \pi_0^j\left[G_j(q_j^{\mathrm{f}*}) - G_j\left(F_j^{-1}\left(\frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j + h_j}\right)\right) - u_j^{\mathrm{f}} F_j^{-1}\left(\frac{p_j + r_j - u_j^{\mathrm{f}}}{p_j + r_j + h_j}\right)\right], \tag{A61}$$

and $VI = \sum_{j=1}^{2} VI^j$. Additionally, using the Leibniz rule and Equation (2) we get

$$\frac{\partial G_j(q^{\mathrm{f}*})}{\partial q_j^{\mathrm{f}*}} = \left[(h_j + p_j + r_j)F_j\left(q_j^{\mathrm{f}*}\right) - p_j - r_j\right].$$

Hence, using Equation (A61) and replacing $Q_j^{\mathrm{f}*}$ from Equation (A59) in addition to replacing π_0^j by $1 - \theta^j + \epsilon^j$ yields:

$$\frac{\partial(VI)}{\partial Q_j^{\mathrm{f}*}} = -\frac{\epsilon^j\left(u_j^{\mathrm{f}} - c^j\right)}{1 - \theta^j}.$$

Therefore, since $u_j^{\mathrm{f}} \geq c^j$ and $\theta^j < 1$ for both $j = 1, 2$, we have $\partial(VI)/\partial q_j^{\mathrm{f}*} \leq 0$ if $\epsilon^j > 0$ and $\partial(VI)/\partial q_j^{\mathrm{f}*} \geq 0$ if $\epsilon^j < 0$ for both $j = 1, 2$. Hence, the proof of the first part is complete. To show the second part (i.e., to show that $VI(\cdot,\cdot)$ is non-decreasing in $y^j = |\epsilon^j|$ for both $j = 1, 2$), notice that:

$$\frac{\partial(VI)}{\partial y^j} = \frac{\partial(VI)}{\partial\epsilon^j}\frac{\partial\epsilon^j}{\partial y^j}.$$

Thus, to show $\partial(VI)/\partial y^j \geq 0$, it is sufficient to show that (for both $j = 1, 2$) $\partial(VI)/\partial\epsilon^j \leq 0$ if $\epsilon^j < 0$ and $\partial(VI)/\partial\epsilon^j \geq 0$ if $\epsilon^j > 0$, which is shown above. ■

Appendix B: Parameter settings

The different parameter settings considered to illustrate the different behaviors, results and insights in Studies A1 and A2 of Appendix C are as follow. In Table A1, setting 2 represents a case with much higher marginal revenues for the products than setting 1. Setting 3 includes changes in other parameters which result in different critical ratios.

The parameter settings considered for Study 1 are listed in Table A2. The first four settings are identical except for the reliability beliefs. The other settings include variations on other parameters as well.

Table A1. Suite of parameter settings in studies A1 and A2

Setting no.	$u_1^f = u_2^f$	j	p_j	r_j	h_j	π_0^j	c^j
1	4.0	1	5.5	5.0	0.5	0.15	3.0
		2	4.0	6.0	0.7	0.12	3.5
2	4.0	1	5.5	15.0	0.5	0.15	3.0
		2	4.0	20.0	0.7	0.12	3.5
3	4.0	1	5.0	8.0	0.5	0.08	3.0
		2	8.0	10.0	0.7	0.03	3.5

Appendix C: Further results on the analysis without recourse

To find the optimal sourcing and contracting levels, let $\tilde{C}(q^1, q^2, q_1^{\mathrm{f}}, q_2^{\mathrm{f}}, \bar{Q}^{\mathrm{f}})$ be the random variable denoting the one-period cost of the firm if it reserves flexible backup capacity $\bar{Q}^{\mathrm{f}}$ and orders q^j $(j = 1, 2)$ units from the dedicated supplier j and q_j^{f} $(j = 1, 2)$ units from the flexible backup supplier for product j. Then, if we let $i^j \in \{0, +\}$ denote the current state of the supplier j, where $i^j = 0$ if supplier j is down and $i^j = +$ otherwise, we have:

$$\begin{aligned} &E_{D_2} E_{D_1}\big[\tilde{C}\big(q^1, q^2, q_1^{\mathrm{f}}, q_2^{\mathrm{f}}, \bar{Q}^{\mathrm{f}}\big)\big] \\ &\quad = u^{\mathrm{f}} \bar{Q}^{\mathrm{f}} + \sum_{j=1}^{2} \big[c_j^{\mathrm{f}} q_j^{\mathrm{f}} + \mathbb{1}_{(i^j \neq 0)}\big(c^j q^j + G_j\big(q^j + q_j^{\mathrm{f}}\big)\big) \\ &\qquad + \mathbb{1}_{(i^j = 0)} G_j\big(q_j^{\mathrm{f}}\big)\big], \end{aligned} \tag{A62}$$

Table A2. Suite of parameter settings in study 1

Setting no.	u^f	j	p_j	r_j	h_j	θ^j	c^j
1	4.0	1	5.5	5.0	0.5	0.80	3.0
		2	4.0	6.0	0.7	0.80	3.5
2	4.0	1	5.5	5.0	0.5	0.85	3.0
		2	4.0	6.0	0.7	0.90	3.5
3	4.0	1	5.5	5.0	0.5	0.90	3.0
		2	4.0	6.0	0.7	0.85	3.5
4	4.0	1	5.5	5.0	0.5	0.95	3.0
		2	4.0	6.0	0.7	0.95	3.5
5	4.0	1	5.5	15.0	0.5	0.85	3.0
		2	4.0	20.0	0.7	0.90	3.5
6	4.2	1	5.0	8.0	0.5	0.85	4.0
		2	8.0	10.0	0.7	0.90	4.0
7	4.5	1	7.0	8.0	0.9	0.95	3.8
		2	8.0	10.0	0.7	0.90	3.5
8	5.0	1	7.0	8.0	0.9	0.85	3.8
		2	8.0	10.0	0.7	0.85	3.5

where $G_j(\cdot)$ is defined in (2). Hence, the expected cost as perceived by the firm is

$$\begin{aligned} &C_{\mathrm{P}}\big(q^1, q^2, q_1^{\mathrm{f}}, q_2^{\mathrm{f}}, \bar{Q}^{\mathrm{f}}\big) \\ &\quad = E_{i^2} E_{i^1} E_{D_2} E_{D_1}\big[\tilde{C}\big(q^1, q^2, q_1^{\mathrm{f}}, q_2^{\mathrm{f}}, \bar{Q}^{\mathrm{f}}\big)\big] \\ &\quad = u^{\mathrm{f}} \bar{Q}^{\mathrm{f}} + \sum_{j=1}^{2} \big[c_j^{\mathrm{f}} q_j^{\mathrm{f}} + \theta^j\big(c^j q^j + G_j\big(q^j + q_j^{\mathrm{f}}\big)\big) \\ &\qquad + (1 - \theta^j) G_j\big(q_j^{\mathrm{f}}\big)\big], \end{aligned} \tag{A63}$$

where the subscript P on $C(\cdot)$ describes that it is the perceived (and not the true) value.

The problem for the firm then is to optimize the ordering and contracting decisions to minimize its *perceived cost* subject to the terms of the contract:

$$\min_{q^1, q^2, q_1^{\mathrm{f}}, q_2^{\mathrm{f}}, \bar{Q}^{\mathrm{f}}} \quad C_P\left(q^1, q^2, q_1^{\mathrm{f}}, q_2^{\mathrm{f}}, \bar{Q}^{\mathrm{f}}\right), \tag{A64}$$

subject to:

$$q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \le \bar{Q}^{\mathrm{f}}, \tag{A65}$$

$$q^j, q_j^{\mathrm{f}} \ge 0 \qquad (j = 1, 2). \tag{A66}$$

Moreover, using Equation (A62), the *true* expected cost of the firm based on any given ordering and contracting decisions can be computed using the true reliabilities as

$$\begin{aligned} &C_{\mathrm{T}}\left(q^1, q^2, q_1^f, q_2^f, \bar{Q}^{\mathrm{f}}\right) \\ &\quad = u^{\mathrm{f}} \bar{Q}^{\mathrm{f}} + \sum_{j=1}^{2} \big[c_j^{\mathrm{f}} q_j^{\mathrm{f}} + (1 - \pi_0^j)\big(c^j q^j + G_j \\ &\qquad \times \big(q^j + q_j^f\big)\big) + \pi_0^j G_j\big(q_j^f\big)\big]. \end{aligned} \tag{A67}$$

Note that while the firm's decisions are based on its perceived reliabilities, the true cost defined in Equation (A67) depends on both the perceived and true reliabilities. In fact, we need to solve model (A64) to (A66) to derive the firm's optimal perceived decisions and implement them in Equation (A67) to determine the associated true total cost.

To solve our non-linear model (A64) to (A66), we first need the following lemma.

Lemma A1. *The objective function* $C_P\left(q^1, q^2, q_1^{\mathrm{f}}, q_2^{\mathrm{f}}, \bar{Q}^{\mathrm{f}}\right)$ *is jointly convex in its variables. Moreover, with* $\theta^j \in (0, 1)$ $(j = 1, 2)$ *and* $u^{\mathrm{f}} \neq 0$*, the convexity is strict.*

Proof of Lemma A1. To show convexity, we need to show that the Hessian matrix (**H**) of the function $C_{\mathrm{P}}(\cdot)$ is positive semi-definite. Using the Leibniz rule, $G_j'(x) = (h_j + p_j + r_j)\, F(x) - (r_j + p_j)$ and $G_j''(x) = (h_j + p_j + r_j)\, f(x)$. Therefore, the Hessian matrix can be written as

$$\mathbf{H} = \begin{bmatrix} a_1 & 0 & a_1 & 0 & 0 \\ 0 & a_2 & 0 & a_2 & 0 \\ a_1 & 0 & b_1 & 0 & 0 \\ 0 & a_2 & 0 & b_2 & 0 \\ 0 & 0 & 0 & 0 & u^{\mathrm{f}} \end{bmatrix},$$

where $a_j = \theta^j\,(h_j + p_j + r_j)\,f_j(q^j + q^{\mathrm{f}}_j)$ and $b_j = a^j + (1-\theta^j)\,f(q^{\mathrm{f}}_j)$ $(j = 1, 2)$. Next, solving the characteristic equation $|\mathbf{H} - \lambda\mathbf{I}| = 0$, we obtain the following eigenvalues:

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}(h_1 + p_1 + r_1)\Big[(1-\theta^1)f_1\big(q^{\mathrm{f}}_1\big) + 2\theta^1 f_1\big(q^1 + q^{\mathrm{f}}_1\big) \\
&\quad + \sqrt{\big[(1-\theta^1)f_1\big(q^{\mathrm{f}}_1\big)\big]^2 + \big[2\theta^1 f_1\big(q^1 + q^{\mathrm{f}}_1\big)\big]^2}\Big],\\
\lambda_2 &= \frac{1}{2}(h_1 + p_1 + r_1)\Big[(1-\theta^1)f_1\big(q^{\mathrm{f}}_1\big) + 2\theta^1 f_1\big(q^1 + q^{\mathrm{f}}_1\big) \\
&\quad - \sqrt{\big[(1-\theta^1)f_1\big(q^{\mathrm{f}}_1\big)\big]^2 + \big[2\theta^1 f_1\big(q^1 + q^{\mathrm{f}}_1\big)\big]^2}\Big],\\
\lambda_3 &= \frac{1}{2}(h_2 + p_2 + r_2)\Big[(1-\theta^2)f_2\big(q^{\mathrm{f}}_2\big) + 2\theta^2 f_2\big(q^2 + q^{\mathrm{f}}_2\big) \\
&\quad + \sqrt{\big[(1-\theta^2)f_2\big(q^{\mathrm{f}}_2\big)\big]^2 + \big[2\theta^2 f_2\big(q^2 + q^{\mathrm{f}}_2\big)\big]^2}\Big],\\
\lambda_4 &= \frac{1}{2}(h_2 + p_2 + r_2)\Big[(1-\theta^2)f_2\big(q^{\mathrm{f}}_2\big) + 2\theta^2 f_2\big(q^2 + q^{\mathrm{f}}_2\big) \\
&\quad - \sqrt{\big[(1-\theta^2)f_2\big(q^{\mathrm{f}}_2\big)\big]^2 + \big[2\theta^2 f_2\big(q^2 + q^{\mathrm{f}}_2\big)\big]^2}\Big],\\
\lambda_5 &= u^{\mathrm{f}}.
\end{aligned}$$

Now notice that since (i) $f_j(\cdot)$ is a probability density function; (ii) $\theta^j \in [0, 1]$; and (iii) $r_j, p_j, h_j, u^{\mathrm{f}} \in [0, +\infty)$, all of the above eigenvalues are non-negative. Therefore, $\mathbf{H}$ is positive semi-definite and hence $C_{\mathrm{P}}(\cdot)$ is jointly convex. Moreover, if $\theta^j \neq 0, 1$ and $u^{\mathrm{f}} \neq 0$, all of the eigenvalues are positive; hence, $\mathbf{H}$ is positive definite. Thus, with $\theta^j \neq 0, 1$ and $u^{\mathrm{f}} \neq 0$, $C_{\mathrm{P}}(\cdot)$ is also *strictly* convex. ■

Study A1. Consider a firm facing two normally (and independently) distributed demands for its products. Particularly, let D_1 and D_2 respectively follow $N(5000, 1200^2)$ and $N(3000, 800^2)$, where $N(\mu, \sigma^2)$ denotes a normal distribution with mean μ and a standard deviation of σ. To illustrate different cases, we consider three sets of different parameter settings presented in Table A2 in Appendix B. Figure A1 depicts the corresponding Improvement Percentages (IP's) in the firm's true expected costs due to contracting with the flexible supplier versus its reliability belief error $\boldsymbol{\Upsilon} = (\epsilon^1, \epsilon^2)$. We denote by $IP_{(F)}\%$ the cost improvement percentage due to the existence of the flexible backup supplier:

$$IP_{(F)}\% = \frac{V^{\mathrm{f}}}{|C_{\mathrm{T}}\,(q'^{1*}, q'^{2*}, 0, 0, 0)|} \times 100.$$

Note that $IP_{(F)}\% > 0$ implies that contracting is profitable and $IP_{(F)}\% \leq 0$ indicates a non-profitable contracting situation. The former case can be seen in parts (a) and (b), and the latter can be seen in part (c) of Figure A1. The corresponding capacity reservation levels for each parameter setting are depicted in Figure A1 (Appendix C). As one specific example, a firm with parameter setting 1 (see Table A1 in Appendix B) and with a reliability belief of $\boldsymbol{\Theta} = (0.8, 0.9)$ (i.e., with belief error $\boldsymbol{\Upsilon} = (-0.05, +0.02)$ based on π^1 and π^2 presented in Table A1) will form a contract and reserve a capacity level of $\bar{Q}^{\mathrm{f}^*} = 6238.9$ units. Based on this decision, the firm would be able to reduce its expected total true costs by $IP_{(F)}\% = 29.5\%$. However, as can be seen in Figure A1 part (a), if this firm has large errors in its reliability belief, it will not be able to greatly reduce its costs by contracting with the flexible backup supplier. (For instance, $IP_{(F)}$ is less than 1% with $\Upsilon \approx (-0.8, -0.8)$.) However, a firm with parameter setting 2 and with the same reliability belief ($\boldsymbol{\Theta} = (0.8, 0.9)$) will reserve a capacity of $\bar{Q}^{\mathrm{f}^*} = 8945.5$ units and will be able to reduce its expected total true costs by $IP_{(F)}\% = 14.6\%$. Although the percentage benefit for this firm is less than the first one, even with large errors in its reliability belief (e.g., with $\boldsymbol{\Upsilon} \approx (-0.8, -0.8)$), as can be seen in Figure A1 part (b), it will still be able to reduce its true expected costs approximately by 14%. In other words, accuracy in estimating the dedicated suppliers' reliabilities is critical for the former firm but not for the latter. This is due to the high profit margins in setting 2 which makes a secondary backup flexible supplier still highly valuable, even with large errors in belief. This results in Observation 2 presented in the main body of the article. ◀

Study A2. Consider a firm facing two normally distributed demands for its products as discussed in Study A1. Let $IP_{(I)}\%$ denote the IP in the firm's expected true costs due to obtaining information about disruption risk of both of its unreliable suppliers. This value can be computed by

$$IP_{(I)}\% = \frac{VI}{|C_{\mathrm{T}}\,(q^{1*}, q^{2*}, Q^{\mathrm{f}*}_1, Q^{\mathrm{f}*}_2, \bar{Q}^{\mathrm{f}*})|} \times 100,$$

where $C_{\mathrm{T}}\,(q^{1*}, q^{2*}, Q^{\mathrm{f}*}_1, Q^{\mathrm{f}*}_2, \bar{Q}^{\mathrm{f}*})$ is the firm's expected true cost under its perceived optimal decisions before obtaining information. Figure A2 illustrates different values of $IP_{(I)}\%$ for the parameter settings 1 to 3 (presented in Table A1 in Appendix B) versus the errors in the firm's reliability belief. As some particular examples, considering the cases discussed in Study A1. A firm with parameter setting 1 and with a reliability belief vector of $\boldsymbol{\Theta} = (0.8, 0.9)$ ($\boldsymbol{\Upsilon} = (-0.05, +0.02)$) can greatly reduce its true costs by $IP_{(I)}\% = 148.9\%$, if it obtains full information about its unreliable suppliers' disruption risk. However, a firm with parameter setting 2 and with same belief (and same error as the previous firm) $\boldsymbol{\Theta} = (0.8, 0.9)$ ($\boldsymbol{\Upsilon} = (-0.05, +0.02)$) can only reduce its cost by $IP_{(I)}\% = 6.9\%$. Finally, a firm with parameter setting 3 and with $\boldsymbol{\Theta} = (0.87, 0.90)$ ($\boldsymbol{\Upsilon} = (-0.05, -0.07)$) can benefit from full information by an amount equal to $IP_{(I)}\% = 19.2\%$ of its current (no-information) true cost. Therefore, if this latter firm is risk-neutral and if establishing a system to obtain such information adds its costs by 10%, it should choose to establish such a system and hence reduce its true expected total cost by 9.2%.

This study reveals additional interesting insights. First, notice that the information is much more valuable in setting

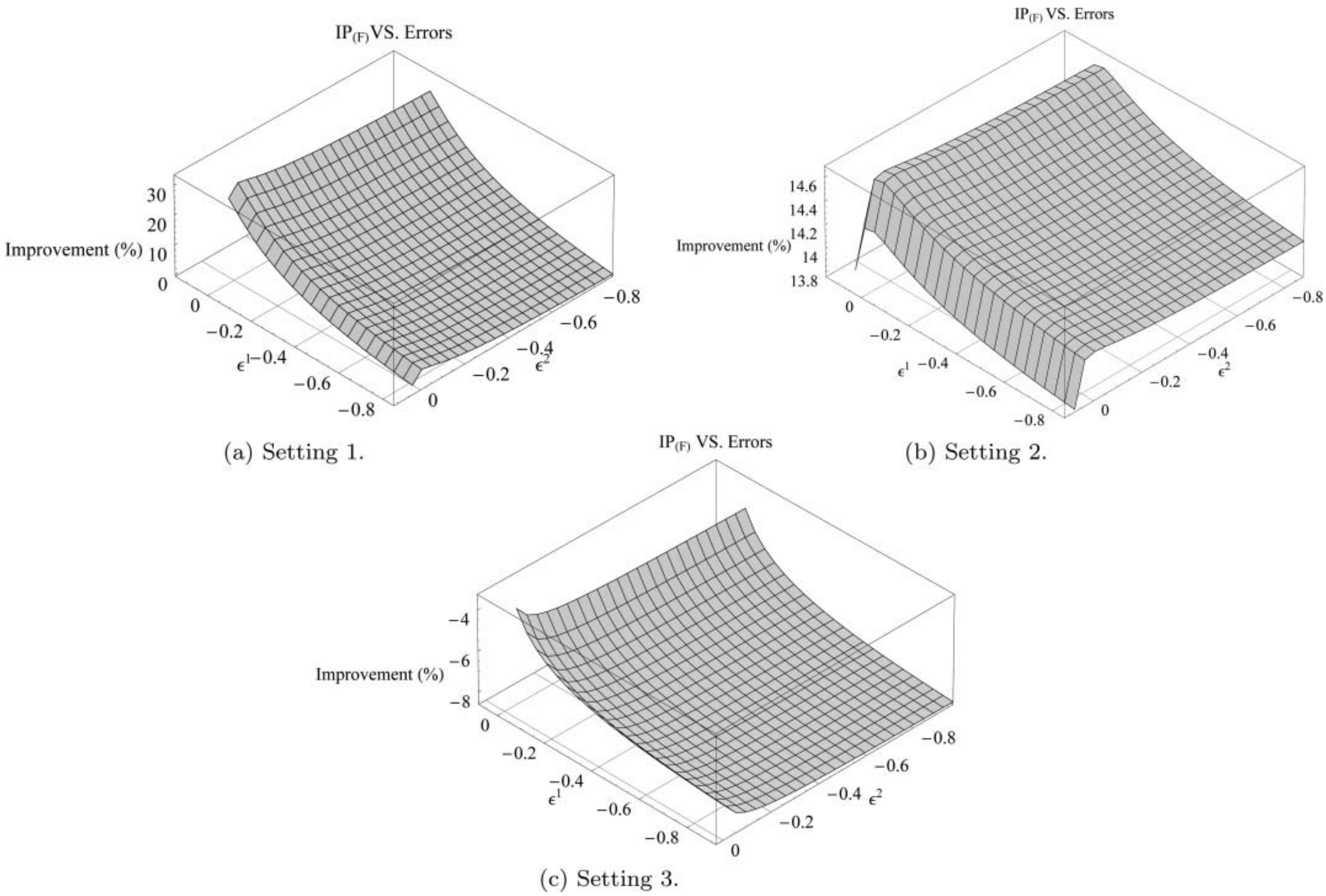


Fig. A1. The value of a flexible backup supplier ($IP_{(F)}\%$) in settings 1 to 3 for different values of error in the firm's reliability belief (ϵ^1, ϵ^2).

1 than setting 2. Since setting 1 represents lower profit margins than setting 2, this suggests Observation 3 presented in the main body of the article.

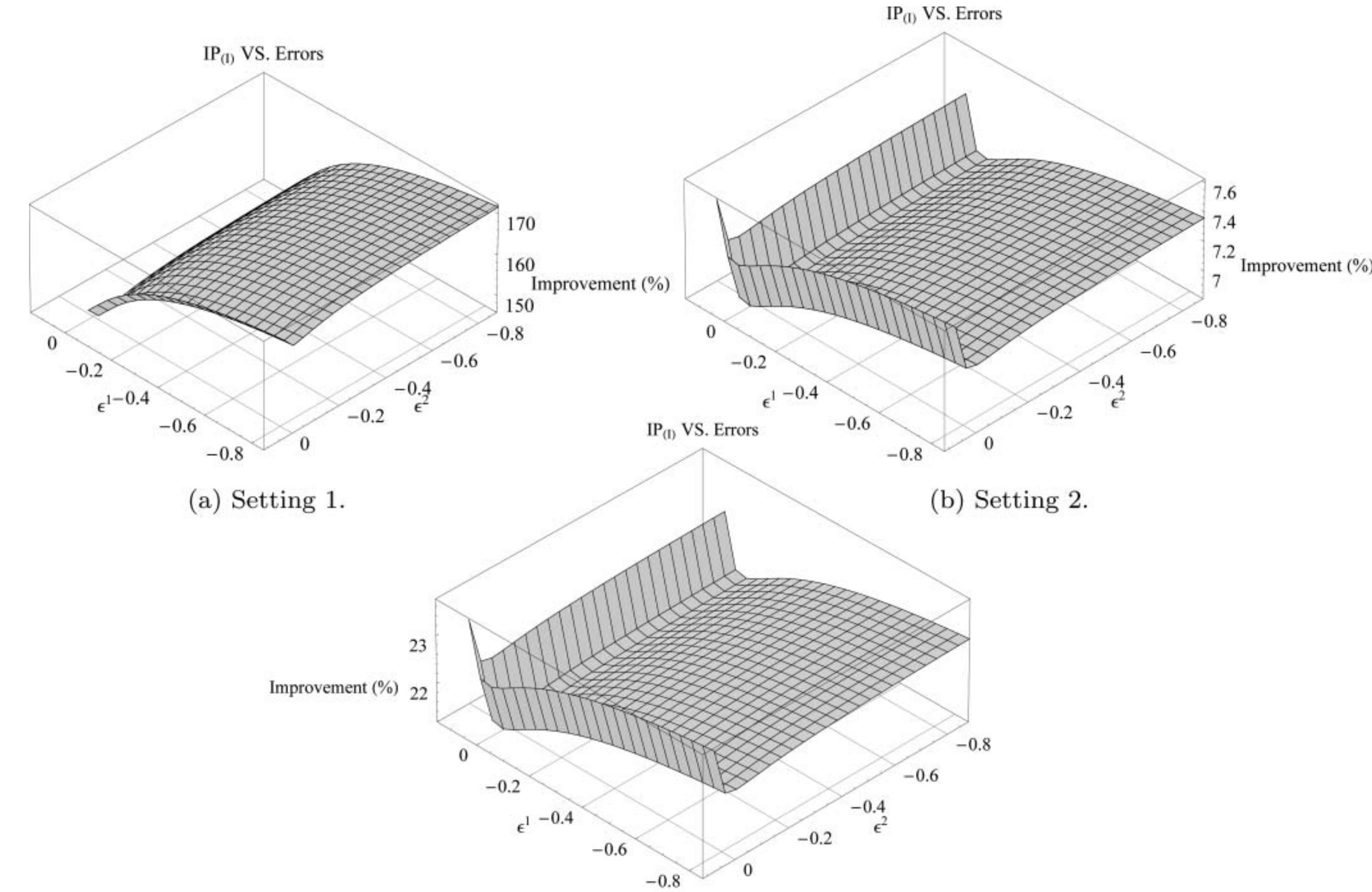
This observation is consistent with what we observed in Study A1 and can be explained as follows. When profit margins are tight, overinvesting in the expensive secondary supplier (resulting from underestimating the reliabilities) is very costly and cannot be justified by the profit obtained from higher potential sales in the case of a disruption. On the other hand, underinvesting in the flexible resource (resulting from overestimating the reliabilities) is more harmful in the case of high profit margins than low ones only when a disruption actually occurs and because of the lost sales. However, the probability of facing a disruption is low. Thus, in expectation, underinvesting is also percentage-wise more costly when profit margins are low than when they are high.

Second, as can be seen in Figure A2, when a firm is overestimating the reliability of its primary supplier (i.e., when $\epsilon^j > 0$), the value of information is much more sensitive to belief errors than the case of underestimating. This results in Observation 4 presented in the main body of the article.

Notice that when a firm overestimates the reliability of its suppliers, it invests less in the secondary flexible backup capacity. In such a situation, with a little better estimation (less error), the firm will invest a little more in the backup capacity. Although the change in investment is relatively small, the cost benefit is large since the backup capacity can work as an *insurance* in the event of pernicious disruptions. ◀

Appendix D: Further results on the two-product setting with recourse

In this appendix, we derive the optimal ordering behavior of the firm (under the two-product setting with recourse) in Stage 2 for the case where $c_j^{\mathrm{f}} > c^j$ for both $j = 1, 2$. Note that the analysis in this case is much simpler than the case $c_j^{\mathrm{f}} \le c^j$ provided in the main body of the article.



(a) Setting 1. (b) Setting 2. (c) Setting 3.

Fig. A2. The value of information ($IP_{(I)}\%$) for different values of firm's reliability belief error (ϵ^1, ϵ^2).

We start with the case where the firm observes both unreliable suppliers to be up.

Proposition A1 (*Both suppliers up*). *Given any reserved flexible backup capacity of $\bar{Q}^{\rm f}$, when $c_j^{\rm f} > c^j$, it is optimal to set*:

$$q_j^{\rm f*} = 0 \quad and \quad q^{j*} = F^{-1}\left(\frac{p_j + r_j - c^j}{p_j + r_j + h_j}\right),$$

when both unreliable suppliers are observed to be up.

When the firm only observes one of the unreliable suppliers to be up, the optimal ordering policy is as follows.

Proposition A2 (*One supplier up*). *Let $m \in \{1, 2\}$ denote the dedicated supplier that is observed to be up, and $n = 3 - m$ be the disrupted supplier. Given a reserved flexible backup capacity of $\bar{Q}^{\rm f}$, when $c_j^{\rm f} > c^j$, it is optimal to set*:

$$q_n^{\rm f*} = \min\left\{\bar{Q}^{\rm f}, F^{-1}\left(\frac{p_j + r_j - c_n^{\rm f}}{p_j + r_j + h_j}\right)\right\}, \quad q_m^{\rm f*} = 0, \quad and$$
$$q^{m*} = F^{-1}\left(\frac{p_j + r_j - c^m}{p_j + r_j + h_j}\right).$$

When the firm observes both unreliable suppliers to be disrupted, the ordering policy is the same as Theorem 3. From these results we have the following:

Observation 10. Unlike the case studied in the main body, when $c_j^{\rm f} \geq c^j$, rationing the limited backup capacity only occurs when both unreliable suppliers are observed to be down.

Proof of Proposition A1. Please note that, as before, the KKT conditions are necessary and sufficient to characterize the optimal solution. Moreover, the KKT conditions are the same as those in the proof of Theorem 1:

$$q_1^{\rm f} + q_2^{\rm f} \leq \bar{Q}^{\rm f}, \tag{A68}$$
$$c^j - \lambda^j = \mu - \lambda_j^{\rm f} + c_j^{\rm f} \qquad (j = 1, 2), \tag{A69}$$
$$q^j + q_j^{\rm f} = F_j^{-1}\left(\frac{p_j + r_j - (c^j - \lambda^j)}{p_j + r_j + h_j}\right) \qquad (j = 1, 2), \tag{A70}$$
$$\mu\left(q_1^{\rm f} + q_2^{\rm f} - \bar{Q}^{\rm f}\right) = 0, \tag{A71}$$

$$\lambda^j q^j = 0 \qquad (j = 1, 2), \tag{A72}$$

$$\lambda_j^{\mathrm{f}} q_j^{\mathrm{f}} = 0 \qquad (j = 1, 2), \tag{A73}$$

$$q^j, q_j^{\mathrm{f}}, \mu, \lambda^j, \lambda_j^f \geq 0. \qquad (j = 1, 2). \tag{A74}$$

Observe that setting $\lambda^j = 0$, $\mu = 0$, $\lambda_j^{\mathrm{f}} = c_j^{\mathrm{f}} - c^j$, $Q_j^{\mathrm{f}*} = 0$, and $q^{j*} = F^{-1}((p_j + r_j - c^j)/(p_j + r_j + h_j))$ satisfy the above KKT conditions. ■

Proof of Proposition A2. Note that the KKT conditions for this case are the same as those in the proof of Theorem 2:

$$q_1^{\mathrm{f}} + q_2^{\mathrm{f}} \leq \bar{Q}^{\mathrm{f}}, \tag{A75}$$

$$c^m - \lambda^m = \mu - \lambda_m^{\mathrm{f}} + c_m^{\mathrm{f}}, \tag{A76}$$

$$q^m + q_m^{\mathrm{f}} = F_m^{-1}\left(\frac{p_m + r_m - \left(\mu - \lambda_m^{\mathrm{f}} + c_m^{\mathrm{f}}\right)}{p_m + r_m + h_m}\right), \tag{A77}$$

$$q_n^{\mathrm{f}} = F_n^{-1}\left(\frac{p_n + r_n - \left(\mu - \lambda_n^{\mathrm{f}} + c_n^{\mathrm{f}}\right)}{p_n + r_n + h_n}\right), \tag{A78}$$

$$\lambda^m q^m = 0, \tag{A79}$$

$$\lambda_m^{\mathrm{f}} q_m^{\mathrm{f}} = 0, \tag{A80}$$

$$\lambda_n^{\mathrm{f}} q_n^f = 0, \tag{A81}$$

$$q^m, q_m^{\mathrm{f}}, q_n^{\mathrm{f}}, \mu, \lambda^m, \lambda_m^f, \lambda_n^f \geq 0. \tag{A82}$$

If

$$\bar{Q}^{\mathrm{f}} \geq F^{-1}\left(\frac{p_j + r_j - c_n^{\mathrm{f}}}{p_j + r_j + h_j}\right),$$

set $\mu = 0$. Otherwise, chose μ such that:

$$F^{-1}\left(\frac{p_j + r_j - c_n^{\mathrm{f}} - \mu}{p_j + r_j + h_j}\right) = \bar{Q}^{\mathrm{f}}.$$

Next, observe that setting $\lambda_m^{\mathrm{f}} = \mu + c_m^{\mathrm{f}} - c^m$, $\lambda_n^{\mathrm{f}} = 0$, $\lambda^m = 0$, $q_n^{\mathrm{f}*} = \min\{\bar{Q}^{\mathrm{f}}, F^{-1}((p_j + r_j - c_n^{\mathrm{f}})/(p_j + r_j + h_j)),$ $q_m^{\mathrm{f}*} = 0$, and $q^{m*} = F^{-1}((p_j + r_j - c^m)/(p_j + r_j + h_j))$ satisfy the KKT conditions, which are sufficient and necessary to characterize the optimal solution. ■

Biographies

Soroush Saghafian has a Ph.D. in Industrial and Operations Engineering from the University of Michigan. His research focus is on the application and development of operations research methods in modeling and control of stochastic systems with specific applications in (i) control of flexible queuing systems, (ii) healthcare operations, and (iii) supply chain and operations management. He has been awarded the 2011 University of Michigan College of Engineering Outstanding Ph.D. Research Award, 2010 INFORMS Pierskalla Award for the best research paper in healthcare (from the Healthcare Applications Society of INFORMS), POMS College of Healthcare Operations Management best student paper award (second place), the 2010 Murty Prize for best research paper in Optimization, and the 2007 IOE Bonder Fellowship award for applied Operations Research. He has also been a finalist for INFORMS 2011 Doing Good with Good OR Award as well as the best student paper award of the Production and Operations Management Society College of Supply Chain in 2009. At the University of Michigan, he taught IOE 440 (Operations Analyses and Management) as the primary instructor in both 2009 and 2010. Prior to joining the University of Michigan, he taught courses in Applied Probability Theory and Plant Layout as a primary instructor. He has served as a referee for various journals, including *Operations Research*, *Operations Research Letters*, *Naval Research Logistics*, *IIE Transactions*, *IEEE Transactions on Evolutionary Computation*, and *Production and Operations Management*.

Mark P. Van Oyen has served as an Associate Professor of Industrial and Operations Engineering at the University of Michigan since 2005. His core interests focus on the analysis, design, control, and management of operations systems, with emphasis on healthcare, service operations, and supply chains and how they can be designed for greater performance, flexibility, and resilience. His research also contributes to applied probability and the control and performance analysis of queuing networks. His awards include IOE Department Faculty of the Year for 2008–2009, ALCOA Manufacturing Systems Faculty Fellow, a best paper award from *IIE Transactions*, and Researcher of the Year from Loyola University Chicago's School of Business. He has served as Associate Editor for *Operations Research*, *Naval Research Logistics*, and *IIE Transactions* and Senior Editor for *Flexible Services & Manufacturing*. He was a faculty member of the Northwestern University School of Engineering (1993–2005) and Loyola University of Chicago's School of Bussiness Administration (1999–2005). He has received grant funding from the National Science Foundation, Office of Naval Research, EPRI, ALCOA, General Motors, and the VA. In industry, he was a researcher with GE Corporate R&D as well as an analysis and simulation engineer with Lear Siegler's Instrument & Avionic Systems Division.